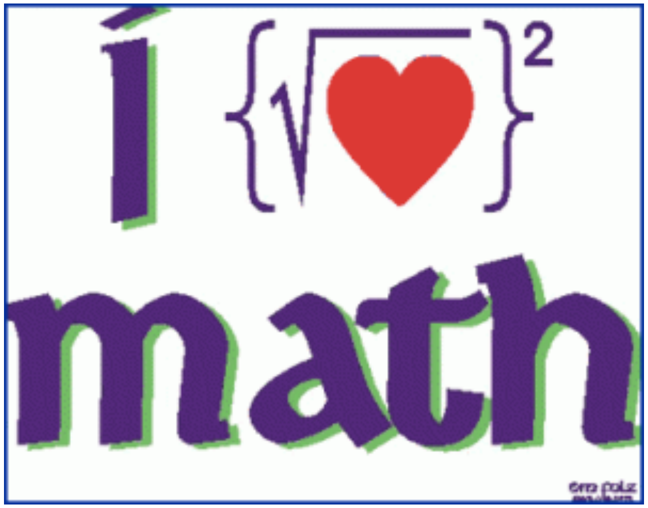

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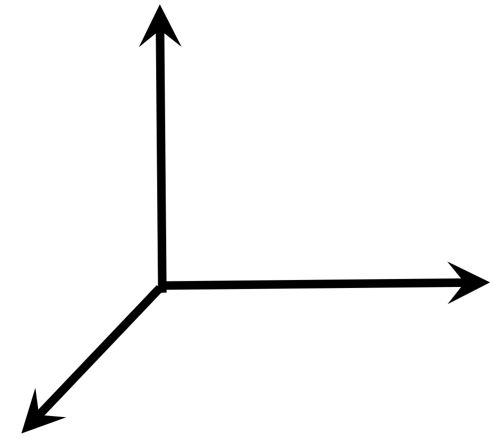


$$\mathbf{v} =$$

$$(\mathbf{v} \cdot \widehat{\boldsymbol{x}})\widehat{\boldsymbol{x}} + (\mathbf{v} \cdot \widehat{\boldsymbol{y}})\widehat{\boldsymbol{y}} + (\mathbf{v} \cdot \hat{\mathbf{z}})\hat{\mathbf{z}}$$

Brook Taylor (1685 – 1731)

Leonhard Euler (1707 – 1783)

Taylor expansion:: $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$

Taylor expansion:: $$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

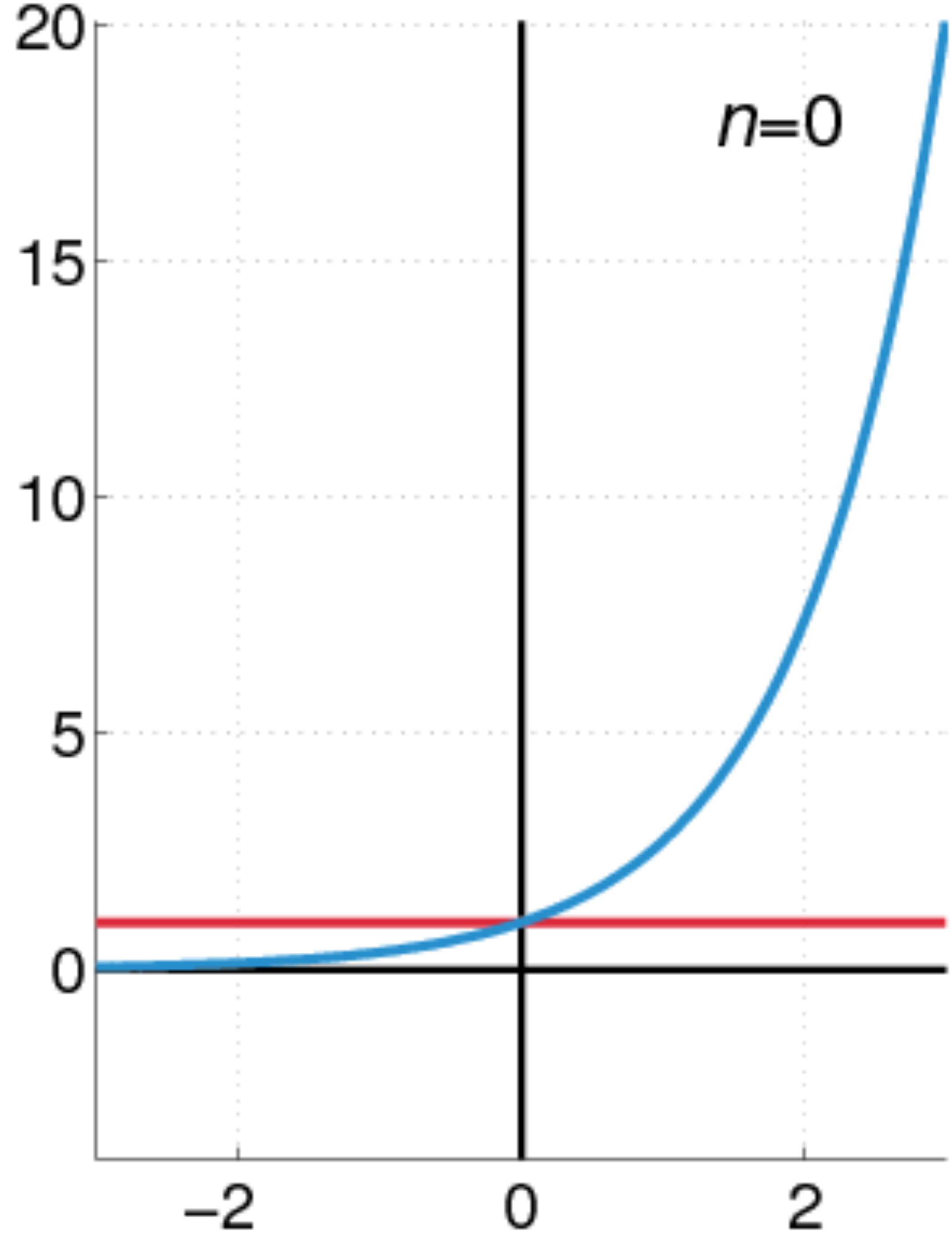
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Euler's formula: $$e^{ix} = \cos x + i \sin x$$

Euler's identity:

$$e^{i\pi} + 1 = 0$$

e: Euler's constant ($\approx$2.71828)
i: imaginary unit of complex numbers
π: pi ($\approx$3.14159)



n=0
20
15
10
5
0
−2
0
2

Local approximation

Versus

Global approximation

Adrien-Marie Legendre
(1752 –1833)

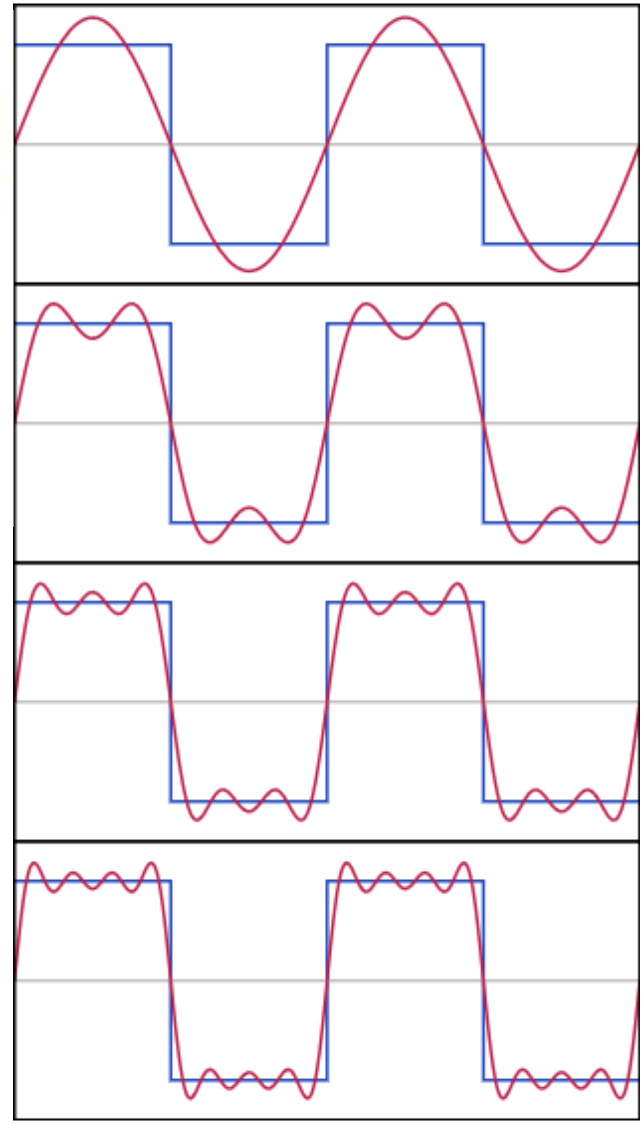
Jean-Baptiste Joseph Fourier (1768 –1830)

Watercolor, 1820

Fourier expansion (series) :

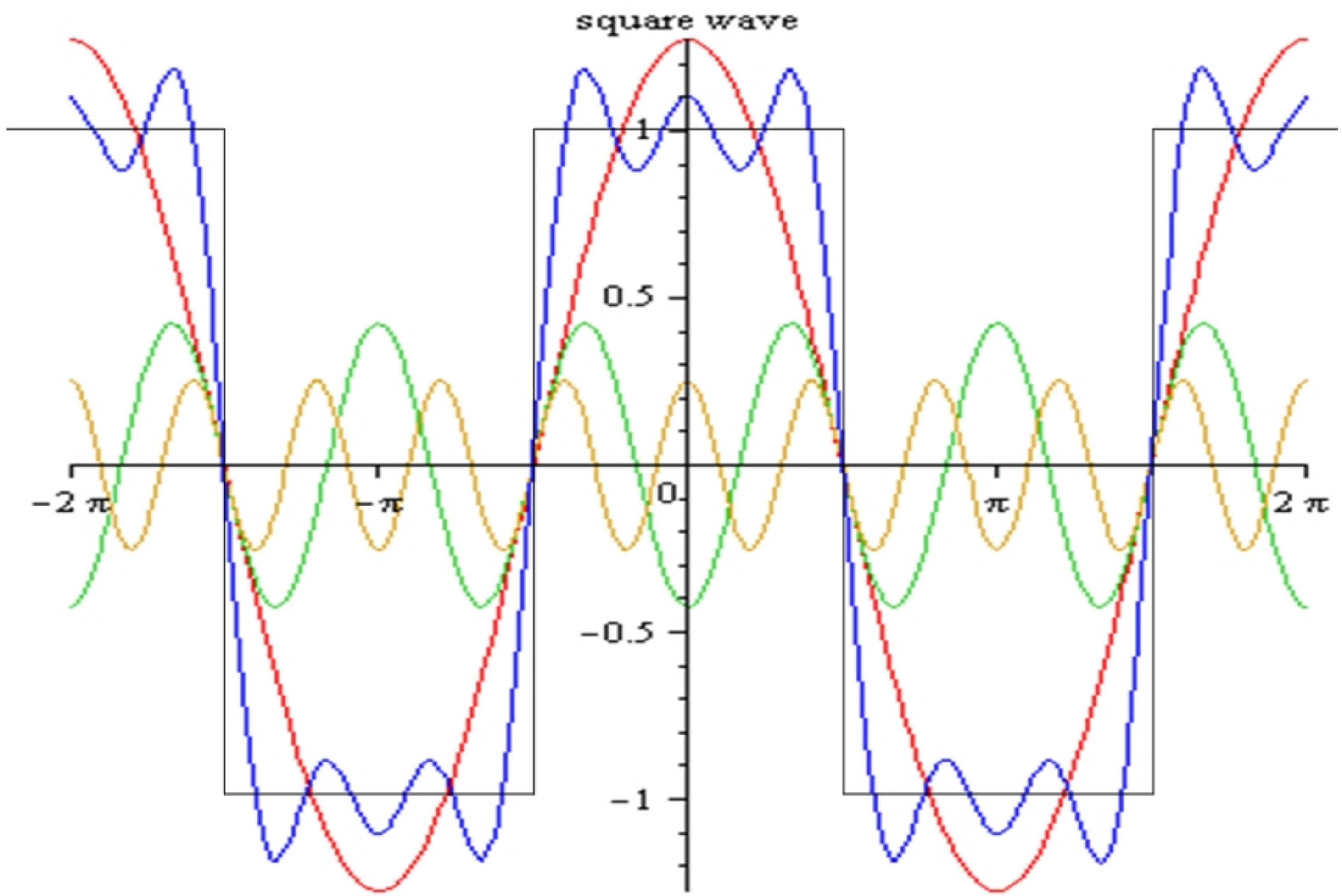
$$g(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$= \sum_{m=0}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$



Fourier transform:

$$\begin{cases} a_0 & = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx, \\ a_n = & = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad 1 \le n \\ b_n & = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \quad 1 \le n. \end{cases}$$



→ **"Spectrum" !!**

Top three most important things
in applied mathematics:

Top three most important things in applied mathematics:

Expansion

Top three most important things
in applied mathematics:

Expansion

Expansion

Top three most important things
in applied mathematics:

Expansion

Expansion

Expansion !

More fun with Fourier….

Parseval's theorem $$\int_{-\infty}^{+\infty} \left(f(t)\right)^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|g(\omega)\right|^2 d\omega$$

Reciprocal space:

Time vs. Frequency

Space vs. Wavenumber

Uncertainty principle (Heisenberg quantum mechanics)

$\Delta f * \Delta T \sim 1$

Δ(mometum) * Δ(distance) ~ h (Planck constant)

Δ(angular momentum) * Δ(angle) ~ h

Δ(energy) * Δ(time) ~ h

Werner Karl Heisenberg (1901 – 1976)

Vector space

Hilbert space

Linear superposition

Basis function set

Orthogonality

Inner product, “Projection”

Completeness

“Complications”, with more fun!

Higher-rank function (scalar, vector, tensor,…)

Multi-variables (space, time,…)

N-dimensional (N=1,2,3,…)

Coordinate system (Cartesian, polar, cylindrical, spherical,… -- be sure it is orthogonal!)

Choices of basis functions (be sure they are orthogonal!)

Surface spherical harmonics

2-D (need 2 indices, degree *n* and order *m*)

Eigen-solution satisfying Laplace equation

→ many wonderful properties

Orthogonal

Span the spherical surface

$$Y_3^{-3}(\theta,\varphi) = \frac{1}{8}\sqrt{\frac{35}{\pi}} \cdot e^{-3i\varphi} \cdot \sin^3\theta \quad = \frac{1}{8}\sqrt{\frac{35}{\pi}} \cdot \frac{(x-iy)^3}{r^3}$$

$$Y_3^{-2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2\theta \cdot \cos\theta \quad = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \cdot \frac{(x-iy)^2 z}{r^3}$$

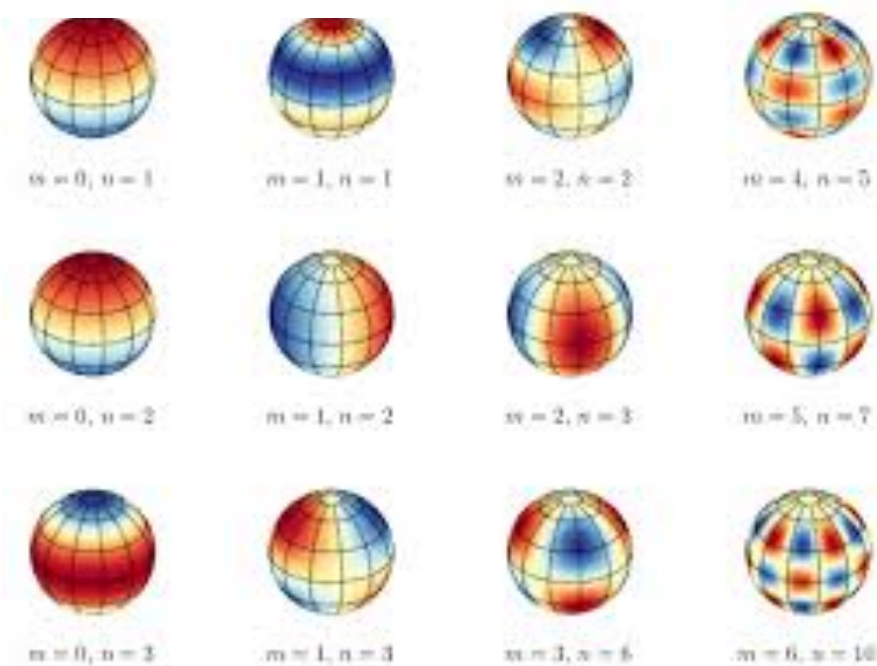
$$Y_3^{-1}(\theta,\varphi) = \frac{1}{8}\sqrt{\frac{21}{\pi}} \cdot e^{-i\varphi} \cdot \sin\theta \cdot (5\cos^2\theta - 1) \quad = \frac{1}{8}\sqrt{\frac{21}{\pi}} \cdot \frac{(x-iy)(4z^2 - x^2 - y^2)}{r^3}$$

$$Y_3^{0}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{7}{\pi}} \cdot (5\cos^3\theta - 3\cos\theta) \quad = \frac{1}{4}\sqrt{\frac{7}{\pi}} \cdot \frac{z(2z^2 - 3x^2 - 3y^2)}{r^3}$$

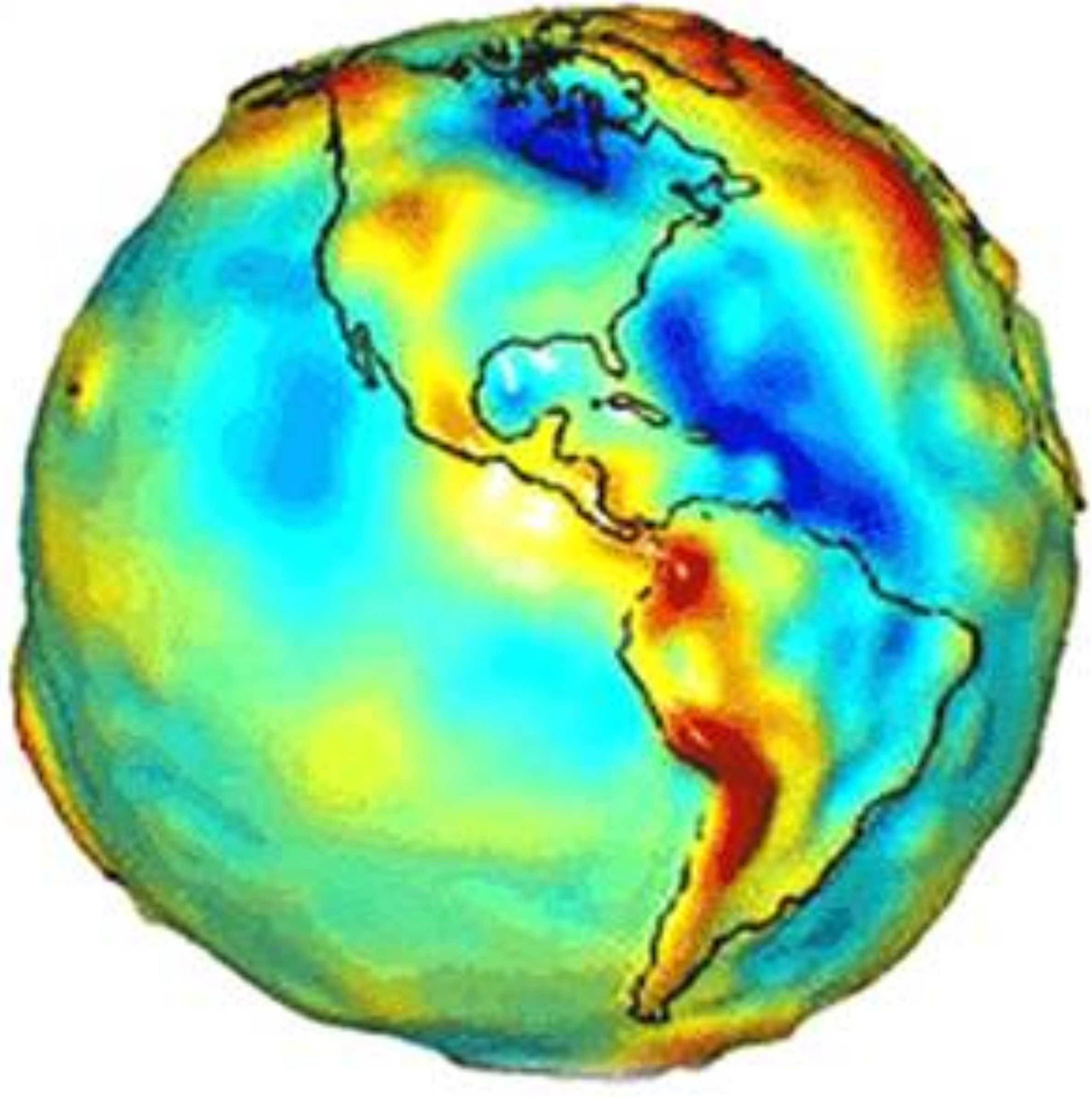
$$Y_3^{1}(\theta,\varphi) = -\frac{1}{8}\sqrt{\frac{21}{\pi}} \cdot e^{i\varphi} \cdot \sin\theta \cdot (5\cos^2\theta - 1) \quad = \frac{-1}{8}\sqrt{\frac{21}{\pi}} \cdot \frac{(x+iy)(4z^2 - x^2 - y^2)}{r^3}$$

$$Y_3^{2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2\theta \cdot \cos\theta \quad = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \cdot \frac{(x+iy)^2 z}{r^3}$$

$$Y_3^{3}(\theta,\varphi) = -\frac{1}{8}\sqrt{\frac{35}{\pi}} \cdot e^{3i\varphi} \cdot \sin^3\theta \quad = \frac{-1}{8}\sqrt{\frac{35}{\pi}} \cdot \frac{(x+iy)^3}{r^3}$$



Earth's Geoid (Static)



Measuring Gravity

-- under the Galilleo-Einstein equivalence principle

- Surface
  - centrifugal (pseudo-)force correction
  - free-air anomaly
  - Bouguer anomaly
- Air-borne, Ship-borne
  - centrifugal (pseudo-)force correction
  - Coriolis (pseudo-)force correction (Eötvös effect)
  - platform acceleration ("noise")

Measuring Gravity

-- under the Galilleo-Einstein equivalence principle

- Space-borne
  - for Earth and planets
  - gravity canceled by centrifugal force (free fall !)

    ⇨ need orbit perturbations, hence orbit tracking 軌道追蹤

“Measuring” Gravity from Space

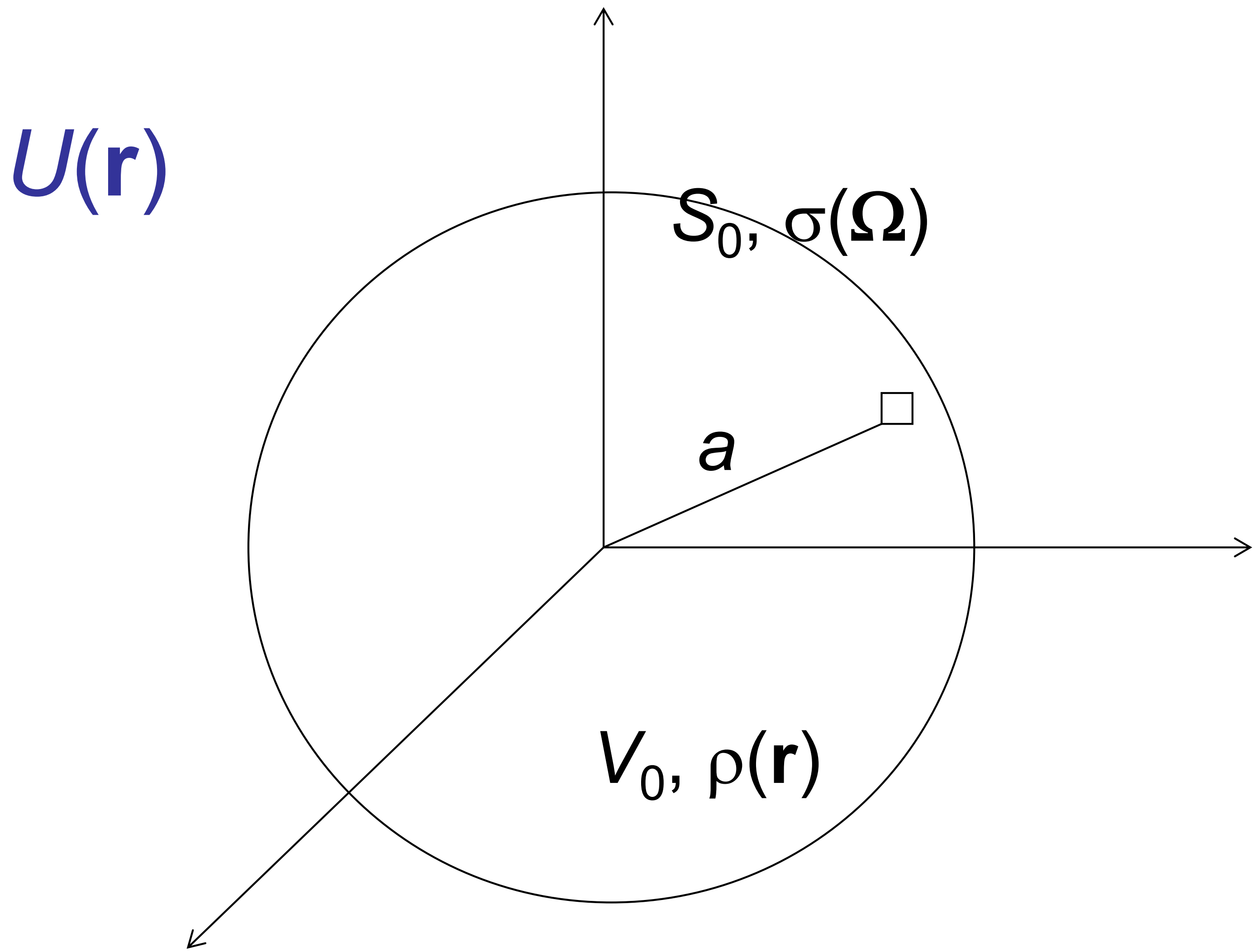
- 1957 Sputnik II – J_2 (Earth oblateness)
- Kaula’s (1966) theory (spherical harmonics)
- Early 1970s, Satellite-laser-ranging
- Spherical harmonic solutions
  - ~1970s, 10 x 10
  - ~1980s, 36 x 36, 70 x 70
  - ~1990s, 180 x 180, 360 x 360
  - “EGM 2008”, degree/order 2159
- time-variable: “J_2 dot” (1980s)
- time-variable: J_2 and low-degree variations (1990s)
- CHAMP (2000), GRACE (2002), GOCE (2009)

Now, **TIME-VARIABLE** gravity field…

(which is 0.0…1% of the gravity field)

new observable!

new data type!



$U(\mathbf{r})$
S_0, $\sigma(\mathbf{\Omega})$
a
V_0, $\rho(\mathbf{r})$

Gravitational Potential Field

- Newton's gravitational law

$$U(\mathbf{r}) = G \iiint_{V_0} \frac{\rho(\mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|} dV_0$$

$$1/|\mathbf{r} - \mathbf{r}_0| = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1)^{-1} (r_0^{\,n} / r^{n+1}) Y_{nm}^{\,*}(\Omega) Y_{nm}(\Omega_0)$$

$\Rightarrow$ Multipole expansion of gravity field

$$U(\mathbf{r}) = G \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{(2n+1)\, r^{n+1}} \left[\iiint_{V_0} \rho(\mathbf{r}_0)\, r_0^{\,n}\, Y_{nm}(\Omega_0)\, dV_0 \right] Y_{nm}^{\,*}(\Omega)$$

Gravitational Potential Field (Geoid)

- Multipole expansion of Newton's formula:

$$U(\mathbf{r}) = G \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{(2n+1)\, r^{n+1}} \left[\iiint_{V_0} \rho(\mathbf{r}_0)\, r_0^{\,n}\; Y_{nm}(\Omega_0)\, dV_0 \right] Y_{nm}^{\;*}(\Omega)$$

- Conventional expression (satisfying Laplace Eq. in terms of Stokes Coeff.):

$$U(r,\theta,\lambda) = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} P_{nm}\,(\mathbf{cos}\,\theta)\left(C_{nm}\, \mathbf{cos}\, m\lambda + S_{nm}\, \mathbf{sin}\, m\lambda\right)$$

$$\Rightarrow \quad \boxed{C_{nm} + iS_{nm} = \frac{1}{(2n+1)\, Ma^{n}} \iiint_{V_0} \rho(\mathbf{r})\, r^{n}\; Y_{nm}(\Omega)\, dV}$$

3-D Gravitational Inversion

- Multipole expansion

$$U(\mathbf{r}) = G \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{(2n+1)\, r^{n+1}} \left[\iiint_{V_0} \rho(\mathbf{r}_0)\, r_0^{\,n}\, Y_{nm}(\Omega_0)\, dV_0 \right] Y_{nm}^{\;*}(\Omega)$$

2n +1 (known) multipoles for each degree n

- Moment expansion

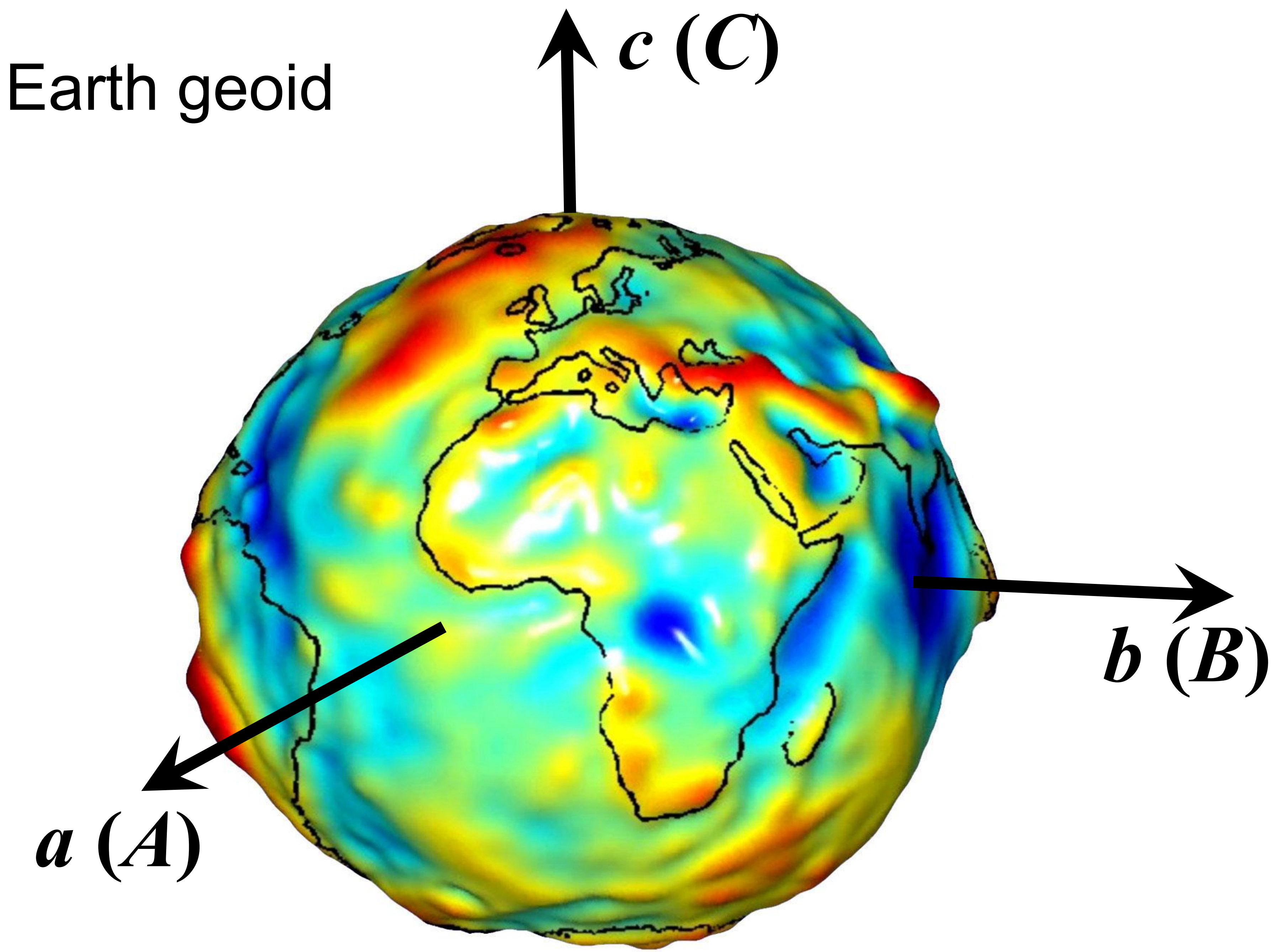
$$U(\mathbf{r}) = G \sum_{n=0}^{\infty} \sum_{\alpha,\beta,\gamma\geq 0}^{\alpha+\beta+\gamma=n} \frac{(-1)^n}{\alpha!\,\beta!\,\gamma!} \left[\iiint_{V_0} x_0^{\alpha}\, y_0^{\beta}\, z_0^{\gamma}\, \rho(\mathbf{r}_0)\, dV_{\,0} \right] \frac{\partial^n}{\partial x^{\alpha} \partial y^{\beta} \partial z^{\gamma}} \left(\frac{1}{|\,\mathbf{r}\,|} \right)$$

(n+1)(n+2)/2 (unknown) moments for each n

$\Rightarrow$ "Degree of deficiency" of knowledge is n(n-1)/2 for each degree n.

Degree n	# multipoles $(2n + 1)$	# moments $(n+1)(n+2)/2$	Degree of deficiency $n(n-1)/2$
0	1 (monopole)	1 (total mass)	0
1	3 (dipole)	3 (center of mass)	0
2	5 (quadrupole)	6 (inertia tensor)	1
3	7 (octupole)	10 (3rd moment)	3
4	9	15	6
5	11	21	10
6	13	28	15
100	201	5151	4950

The degree of deficiency as a function of spherical harmonic degree n in the 3-D gravitational inversion.


Earth geoid
c (C)
b (B)
a (A)

Additional physical/mathematical constraints leading to unique solutions:

- minimum shear energy
- maximum entropy of ρ
- minimum norm-2 variance for the lateral distribution
- …………

2-D gravitational Inversion on a spherical shell S_0

$$C_{nm} + iS_{nm} = \frac{a^2}{(2n+1)M} \iint_{S_0} \sigma(\Omega)\, Y_{nm}(\Omega)\, d\Omega$$

$$\sigma(\Omega) = \Sigma_{n,m} \sigma_{nm} Y^*_{nm}(\Omega)$$

$$\Rightarrow \qquad \sigma_{nm} = \frac{(2n+1)M}{4\pi a^2}\left(C_{nm} + iS_{nm} \right)$$

It is possible to mimic ANY external field by means of some proper surface density on S_0.

2-D gravitational Inversion on a spherical shell S_0 (cont'd)

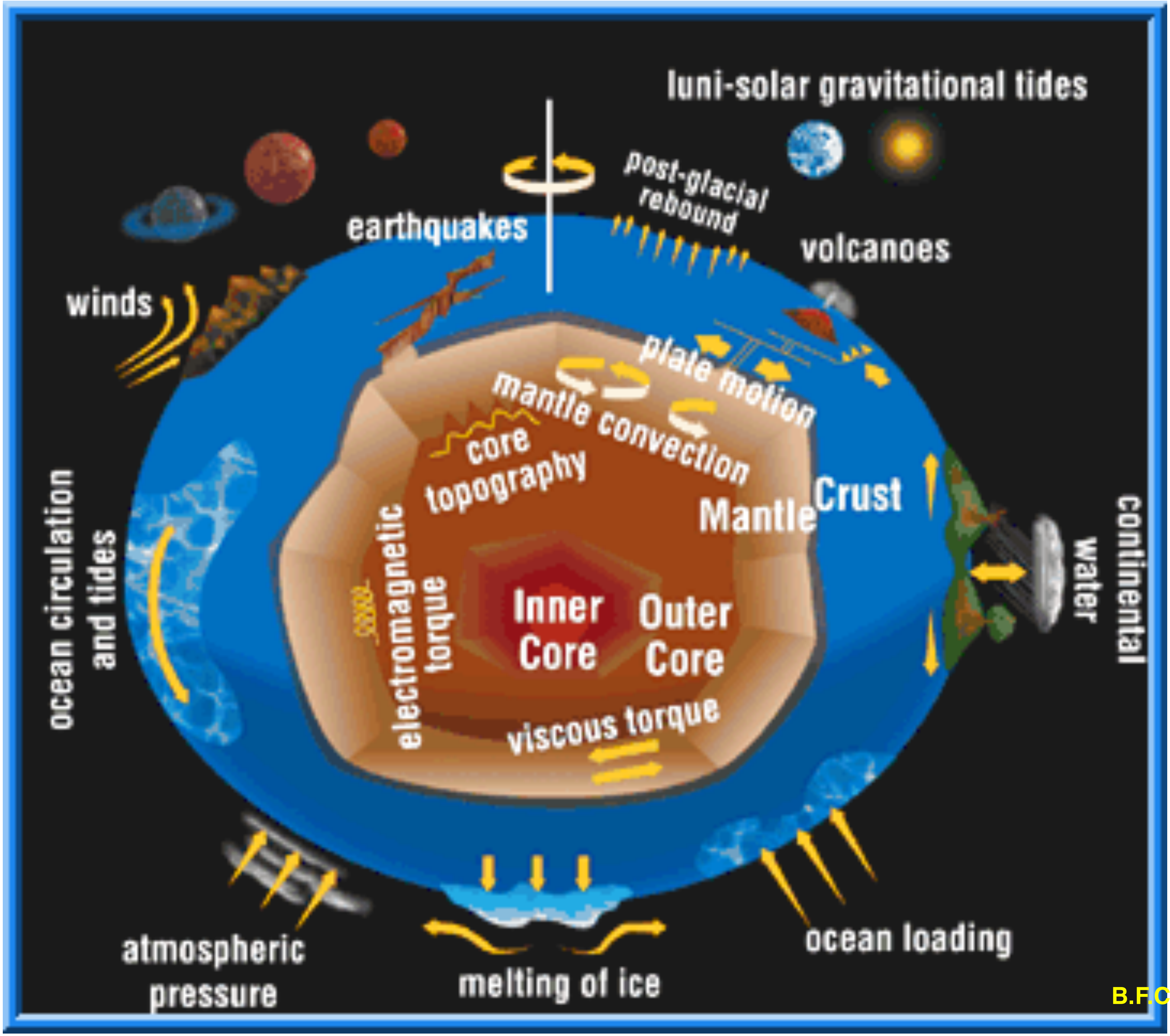
For CHANGES due to mass redistribution on S_0 (taking into account of loading effect), in Eulerian description:

$$\Delta C_{nm}(t) + i\,\Delta S_{nm}(t) = \frac{a^2(1+k_n')}{(2n+1)M} \iint \Delta\sigma(\Omega;t)\, Y_{nm}(\Omega)\, d\Omega$$

$$\Rightarrow \boxed{\Delta\sigma_{nm}(t) = \frac{(2n+1)M}{4\pi a^2(1+k_n')}\left[\Delta C_{nm}(t) + i\,\Delta S_{nm}(t)\right]}$$

Unique!

Loading effect "undone"


luni-solar gravitational tides
post-glacial rebound
earthquakes
volcanoes
winds
plate motion
mantle convection
core topography
Crust
Mantle
ocean circulation and tides
electromagnetic torque
Inner Core
Outer Core
continental water
viscous torque
atmospheric pressure
melting of ice
ocean loading

B.F.Chao

(near) Surface Mass Transports

- Earth ellipticity ~ ½ of 1/300 a ~ 10 km
- Atmosphere scale height ~ 10 km
- Ocean < ~ 5 km
- Land hydrology < a few km
- Crustal/topography change < ~ 30 km

Nice things about spherical harmonics:

- Wavelength, or spatial resolution,

  $\sim 40{,}000/2N$ km $\Longrightarrow$ Concept of spectrum

- Altitude attenuation $\sim r^{-n+1}$
- Geoid: Stokes Coeff. (C_{nm}, S_{nm})
- Gravity Disturbance: $(n+1)*(C_{nm}, S_{nm})$
- Gravity Anomaly: $(n-1)*(C_{nm}, S_{nm})$
- Surface Mass Change: $(2n+1)*(\Delta C_{nm}, \Delta S_{nm})$

Empirical Orthogonal Function (EOF)

1. Making the covariance matrix : $D^T D$

data matrix (D) = $\begin{bmatrix} g(s_1,t_1) & g(s_2,t_1) & \cdots & g(s_i,t_1) \\ g(s_1,t_2) & g(s_2,t_2) & \cdots & g(s_i,t_2) \\ \vdots & \vdots & \ddots & \vdots \\ g(s_1,t_j) & g(s_2,t_j) & \cdots & g(s_i,t_j) \end{bmatrix}$

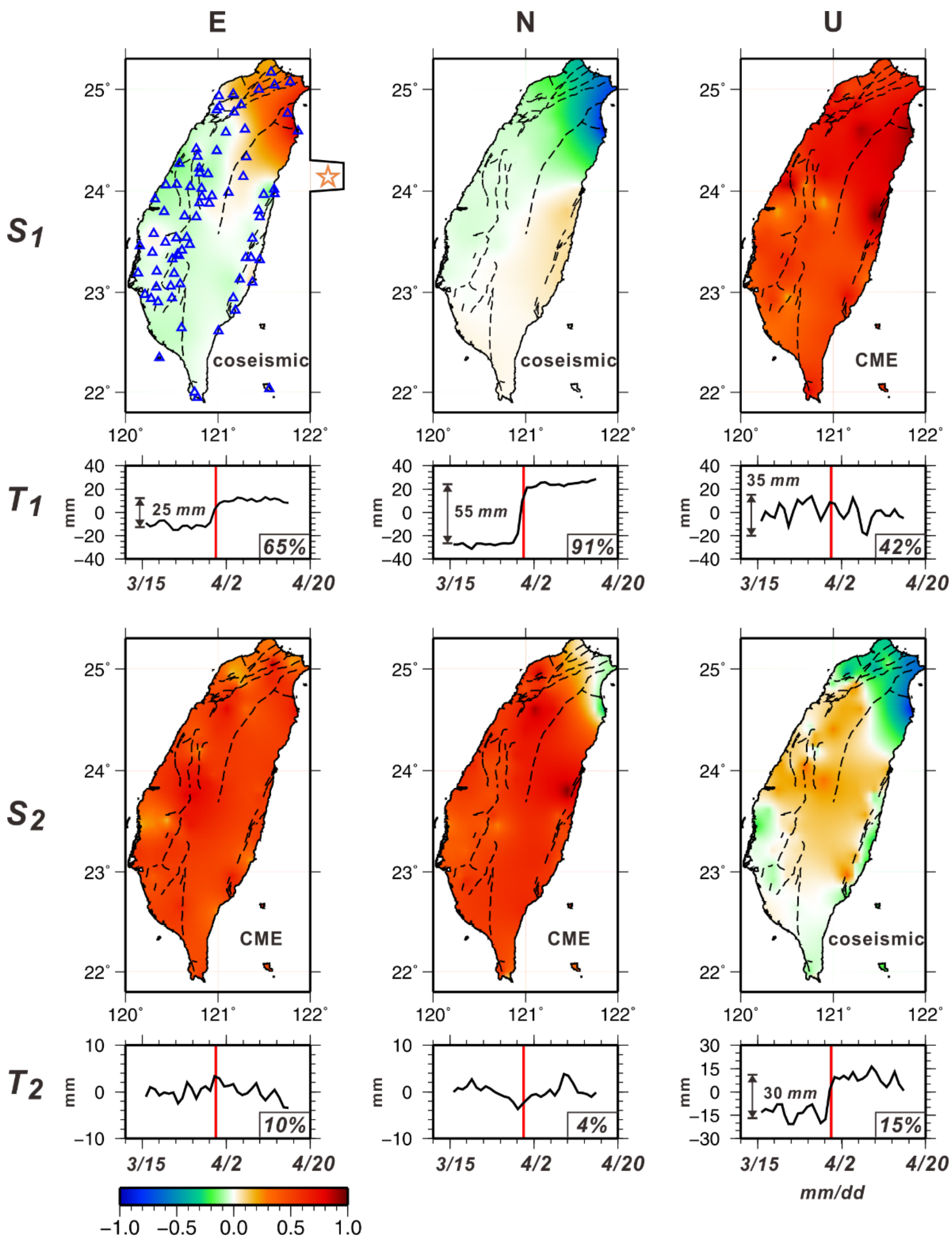
s_i : *i*th location
t_j : *j*th time

2. Decomposing covariance matrix by **eigenvector**.

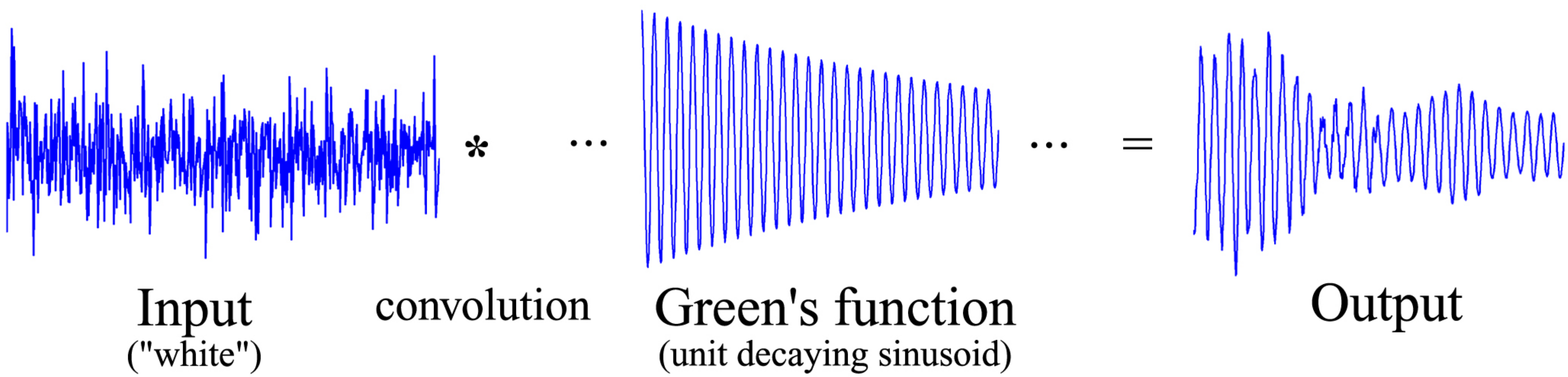
$$g(x,y,t) = \sum_{m=1}^{M} u_m(t) e_m(x,y)$$

$$\Phi_m = \lambda_m \Big/ \sum_{k=1}^{M} \lambda_k$$

$e_m(x,y)$: eigenvector, also is pattern of mode *m*
$u_m(t)$: amplitude (time series) of mode *m*; defined by projecting the data onto the eigenvector.
λ_m : eigenvalue of the eigenvector, $e_m(x,y)$
Φ_m : percentage variance of λ_m in total *M* modes



2002/3/31
earthquake



Input
("white")
*
convolution
...
Green's function
(unit decaying sinusoid)
...
=
Output

A general linear system (a "filter"):

Input → System function → Output
Excitation → Green's function → Observation

System function (= Green's function):

Time domain: Impulse response $G(t)$

Frequency domain: Transfer function $G(\omega)$

$$\text{Observation}(t) = \int_{-\infty}^{t} \text{Excitation}(\tau)\, G(t-\tau)\, d\tau$$

$$\text{Observation}(\omega) = \text{Excitation}(\omega)\, G(\omega)$$

Consider a **resonance** system with (many) normal modes (which are orthogonal or "normal" to each other, and form a complete set) with frequency ω_0's, such as a musical instrument,

$G(t) \sim \exp(i\omega_0)$.

NOW LET IT SPIN

(fundamental changes due to the anti-self-adjoint Coriolis force)

Resonance frequencies (eigen-frequencies) shifted/split

Spatial patterns (eigen-functions) modified

$\Rightarrow$ generalized orthogonality

Excitation formula modified

=> resonance factor changed

from $1/(\omega_0^2 - \omega^2)$ to $1/2\omega_0(\omega_0 - \omega)$.

(Chao, 1982)

Normal-mode decomposition (harmonic response excited by $\mathbf{E}\exp(i\omega t)$):

In non-rotating frame, such as ordinary musical instrument:

$$\mathbf{S}(\mathbf{r},t)=\sum_n \frac{1}{\omega_n^2-\omega^2}(\mathbf{u}_n^*,\mathbf{E})\,\mathbf{u}_n(\mathbf{r})\exp(i\omega t)$$

In rotating frame, such as Erath (with Coriolis force):

$$\mathbf{S}(\mathbf{r},t)=\sum_n \frac{1}{2\omega_n(\omega_n-\omega)}(\mathbf{u}_n^*,\mathbf{E})\,\mathbf{u}_n(\mathbf{r})\exp(i\omega t)$$

(Chao, 1982)

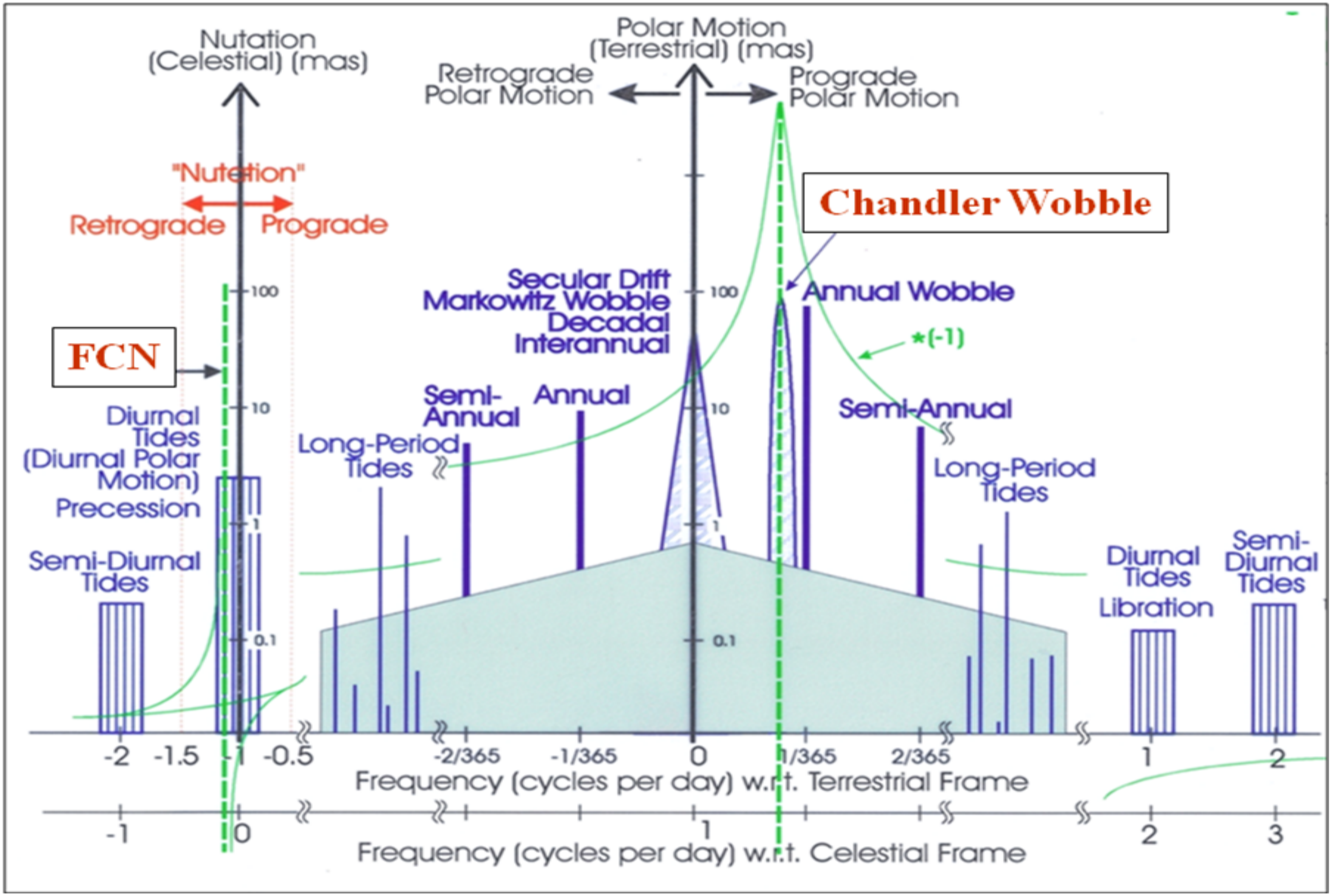
Rigid-body modes

- 6 trivial solutions to the normal mode equation
- For non-rotating Earth, “zero-frequency”:
  - 3 for translation (${}_0S_1$)
  - 3 for rotation (${}_0T_1$)
- For rotating Earth:
  - 3 for translation
  - 3 for rotation (with non-zero frequency), -- axial spin, tilt-over, Euler-Chandler wobble, no longer “normal”, but generalized orthogonal

The Earth has two known intrinsic normal modes of rotation:

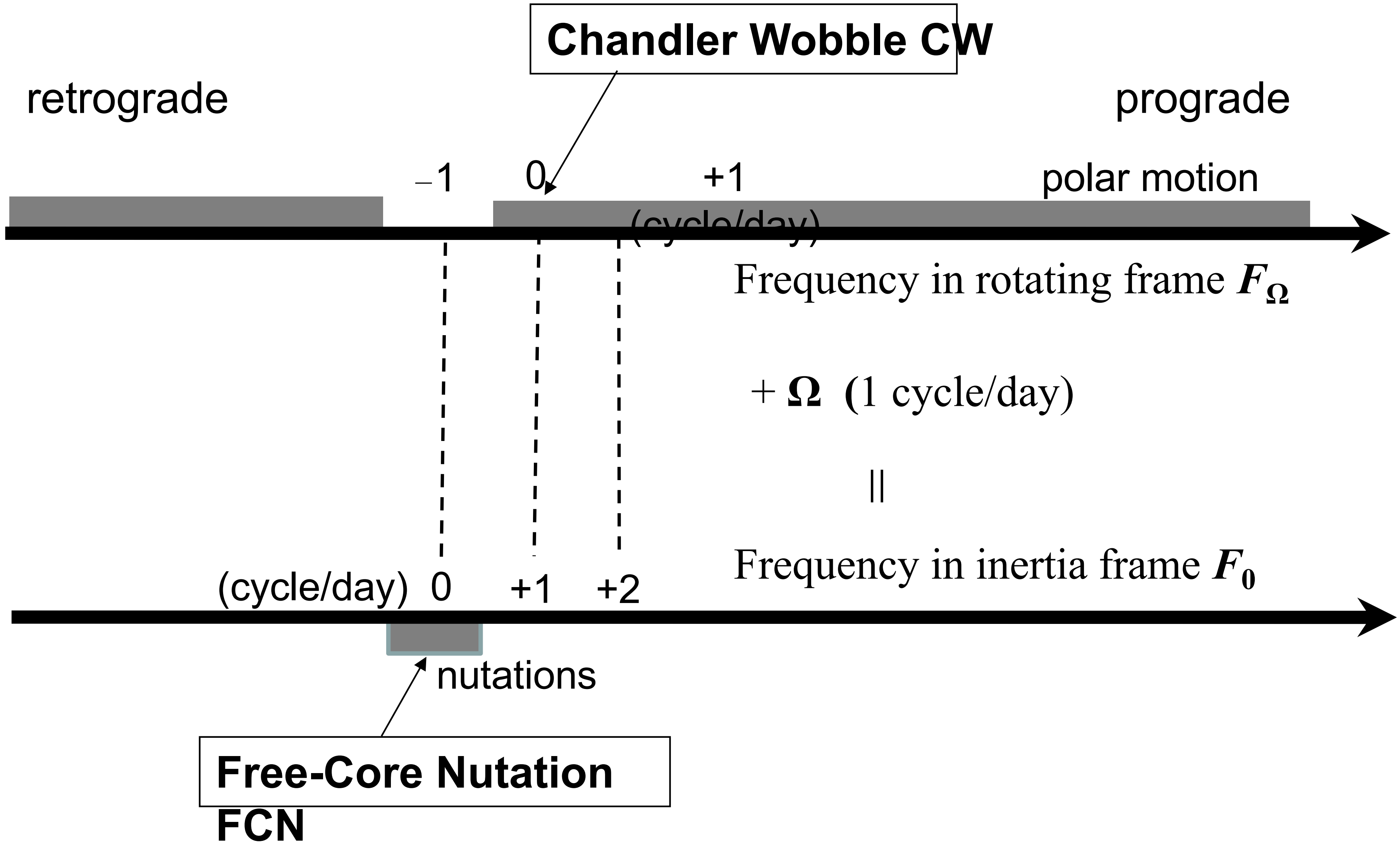
•**Chandler wobble** of the solid mantle has a prograde period in the rotating terrestrial frame of +434 days and a *Q* of ~50;

•**Free core nutation** of the fluid outer core has a retrograde period in the inertia celestial frame of ~—400+ (?) sidereal days and a high *Q*.



$$m'(t)\ [\text{w.r.t. } \boldsymbol{F}_{\boldsymbol{\Omega}}] = -m(t)\ [\text{w.r.t. } \boldsymbol{F}_{\boldsymbol{0}}] \cdot \exp(-i\Omega t).$$

$$\omega'\ [\text{w.r.t. } \boldsymbol{F}_{\boldsymbol{\Omega}}] = \omega\ [\text{w.r.t. } \boldsymbol{F}_{\boldsymbol{0}}] - \Omega$$



Chandler Wobble CW
retrograde
prograde
−1
0
+1
polar motion
(cycle/day)
Frequency in rotating frame F_Ω
$+ \boldsymbol{\Omega}$ (1 cycle/day)
||
Frequency in inertia frame F_0
(cycle/day)
0
+1
+2
nutations
Free-Core Nutation FCN

Excitation of a rotational mode

$$m(t) = \frac{1}{2\Omega(\sigma_0 - \omega)} E \exp(i\omega t) \,, \quad m(t) = x(t) + i\,y(t).$$

for a normal mode excited by a harmonic source in the rotating frame. This equation is the basis for the resonance method of estimating *P* and *Q*.

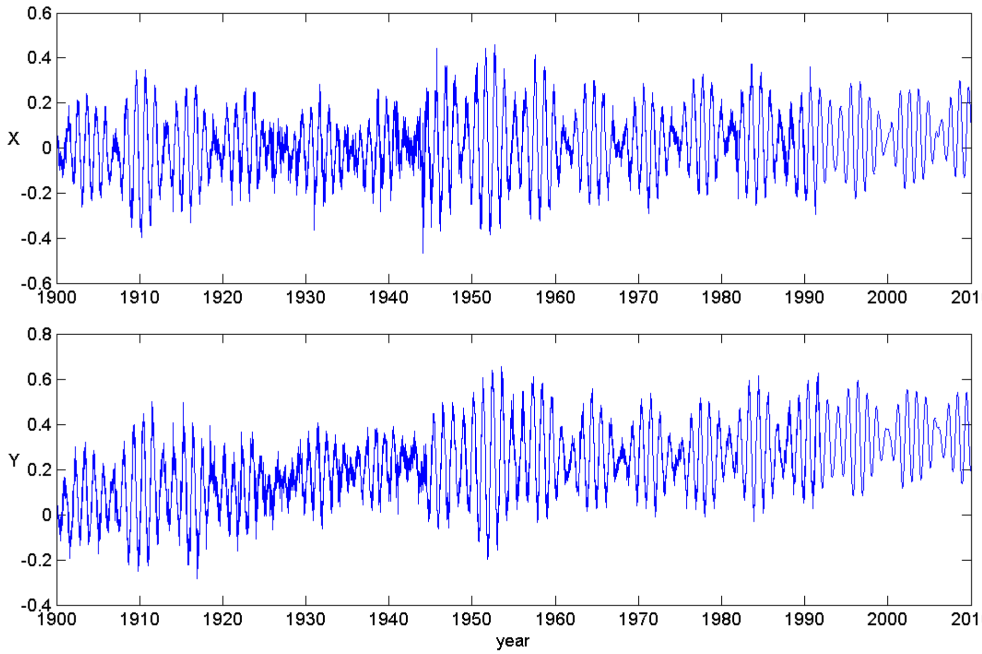
$$m(t) = -\frac{i}{2\Omega} \int_{-\infty}^{t} \exp[i\sigma_0(t-\tau)] E(\tau) d\tau$$

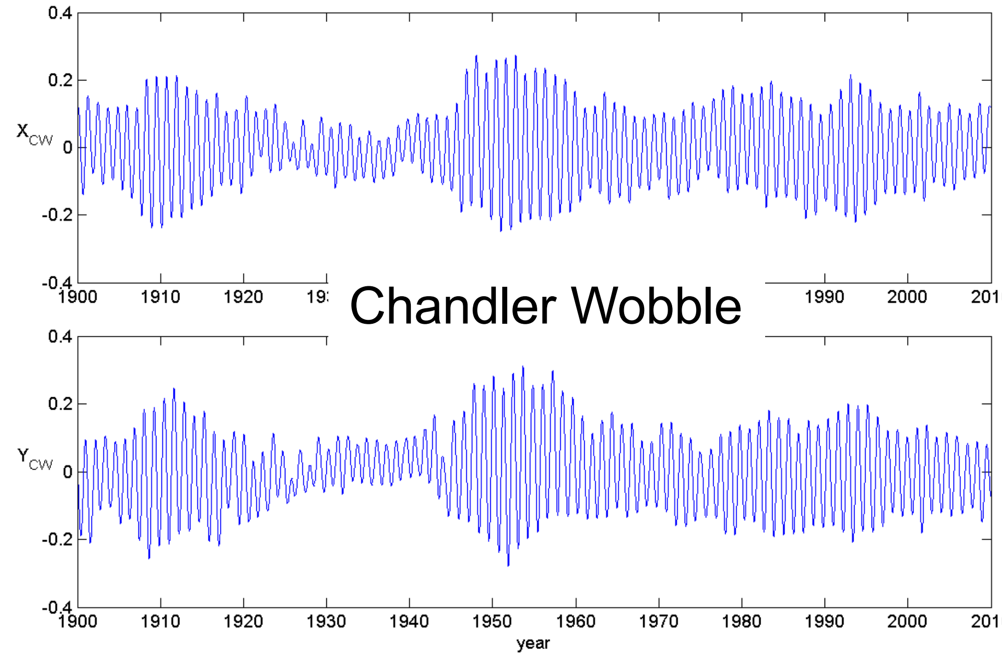
for an arbitrary excitation $E(t)$. This equation of convolution describes the observed motion – fluctuating amplitude, apparent period, and phase.

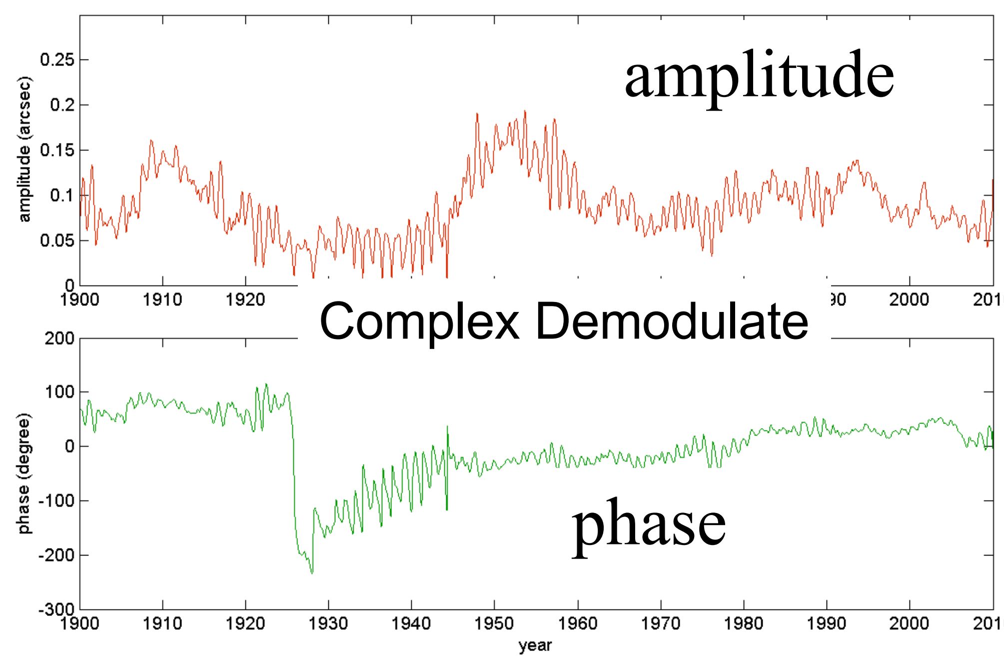
$$\frac{i}{\sigma_0} \frac{dm(t)}{dt} + m(t) = \frac{1}{2\Omega\sigma_0} E(t)$$

as a differential deconvolution form of the excitation equation. This equation is the basis for the deconvolution method of estimating *P* and *Q*.

Polar Motion

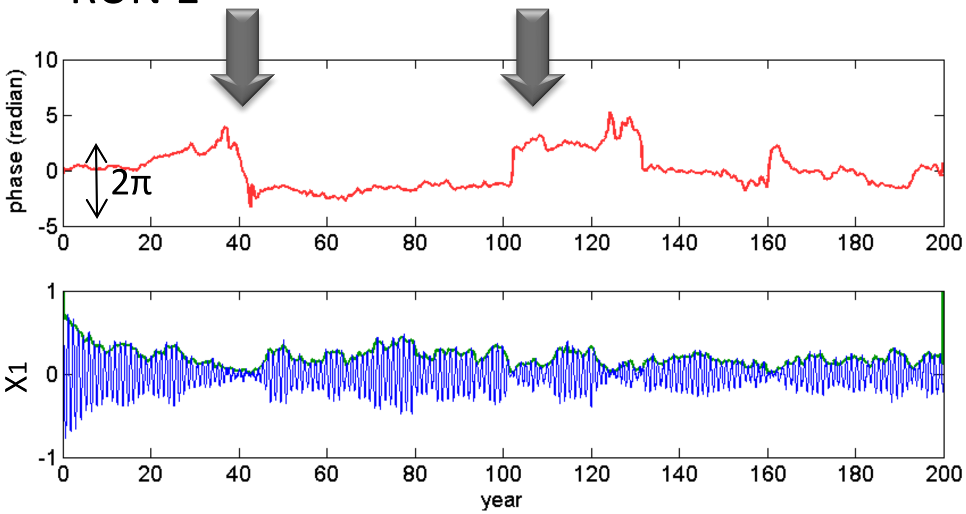




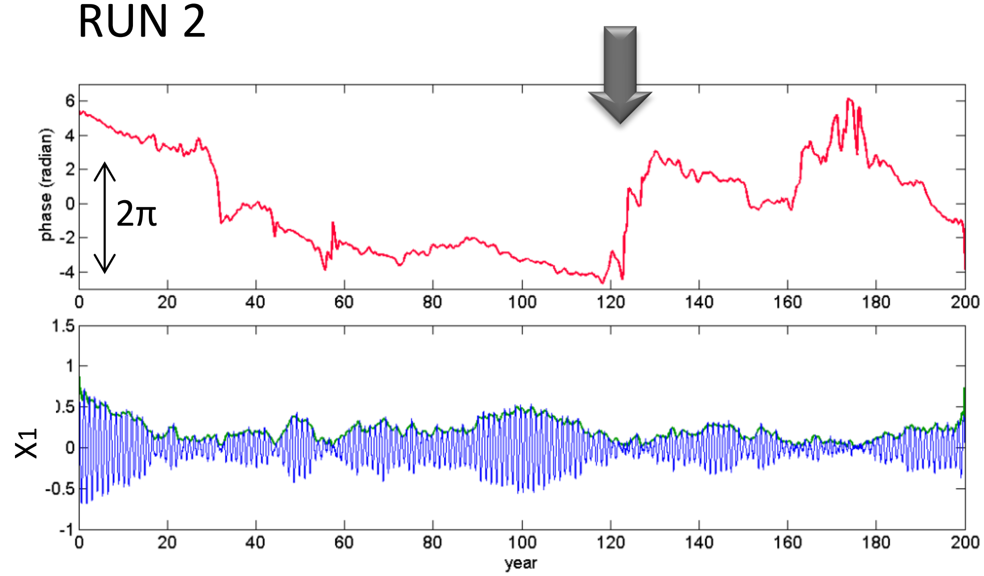


Monte Carlo experiments

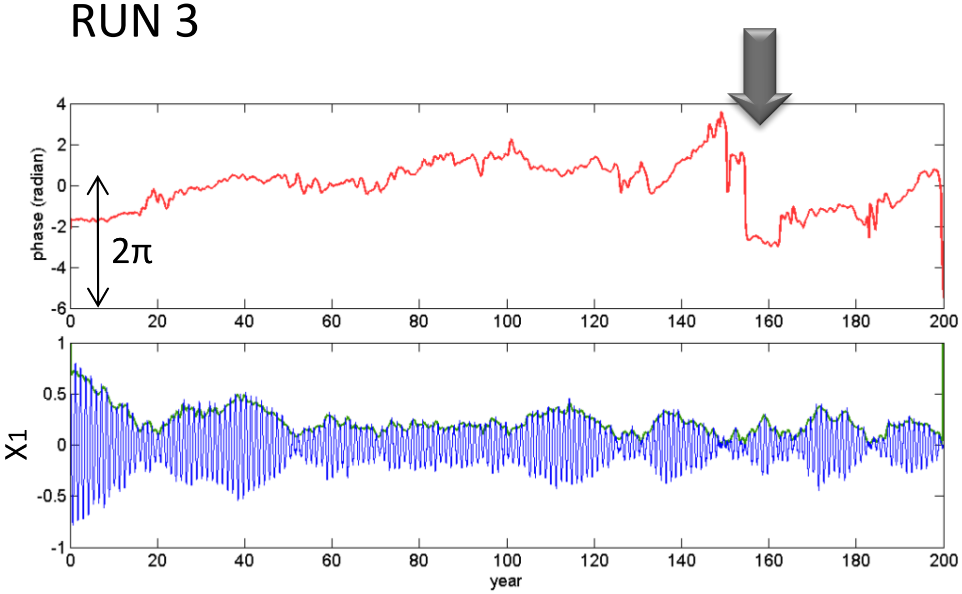
RUN 1



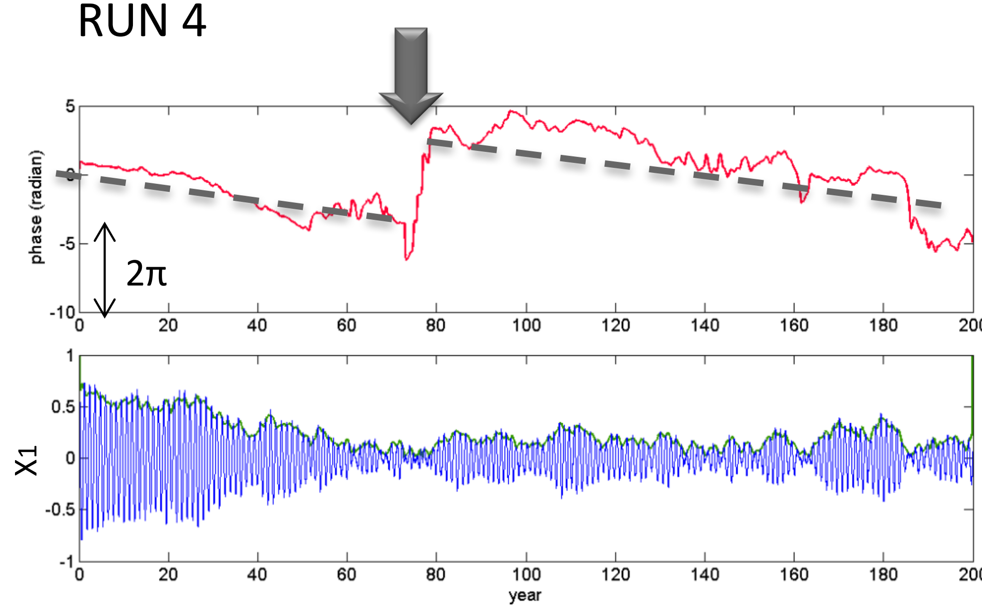
RUN 2



RUN 3

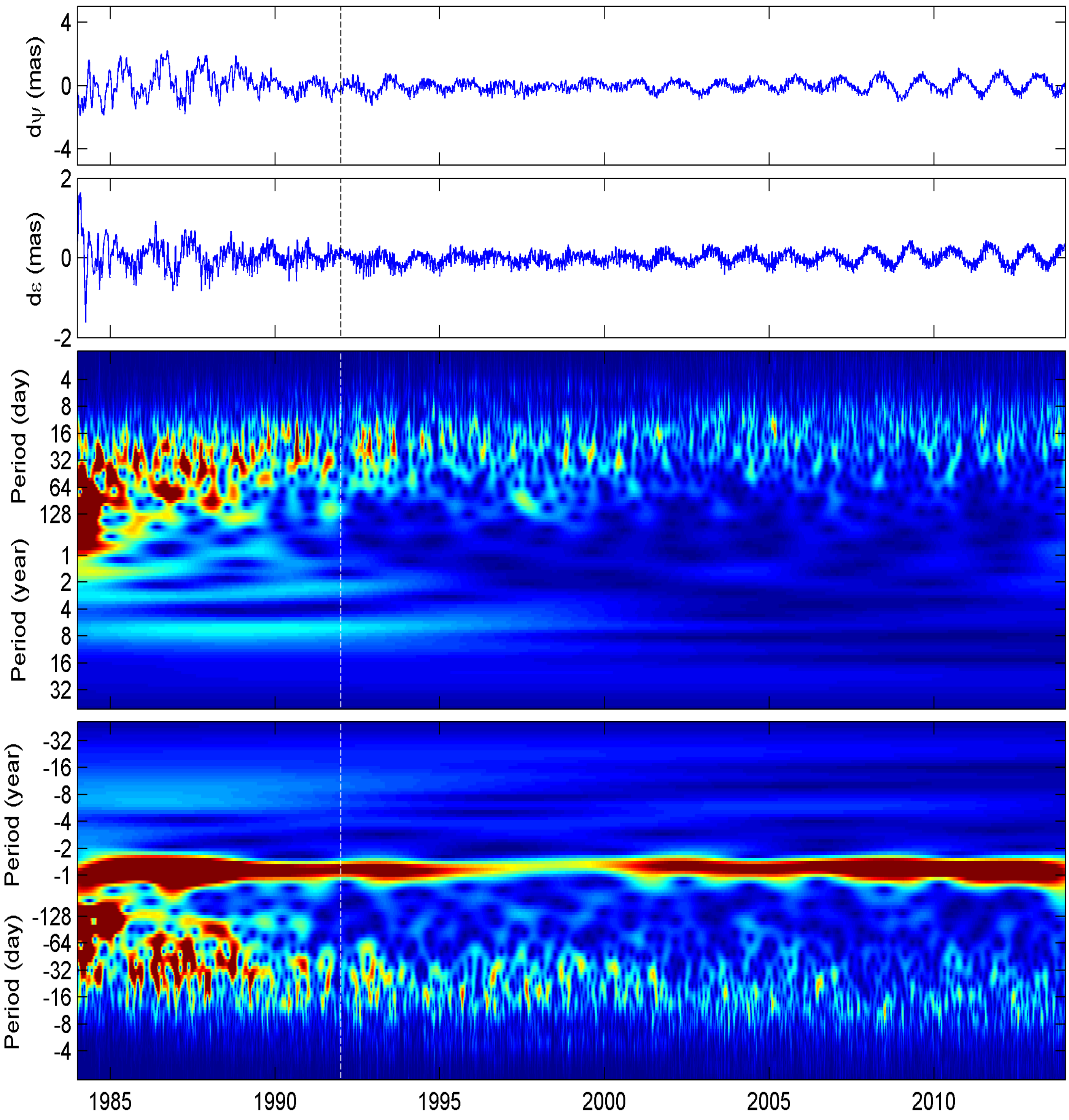


RUN 4



VLBI data for FCN: A quick look

The VLBI celestial pole offsets $d\Psi$ and $d\varepsilon$ data as provided by IERS have been homogenized and slightly smoothed to nominal intervals of 1 solar day. All tides have been removed. We use data spanning 23 years of 1992-2014, as well as 11-year segmented data for various tests and comparisons.

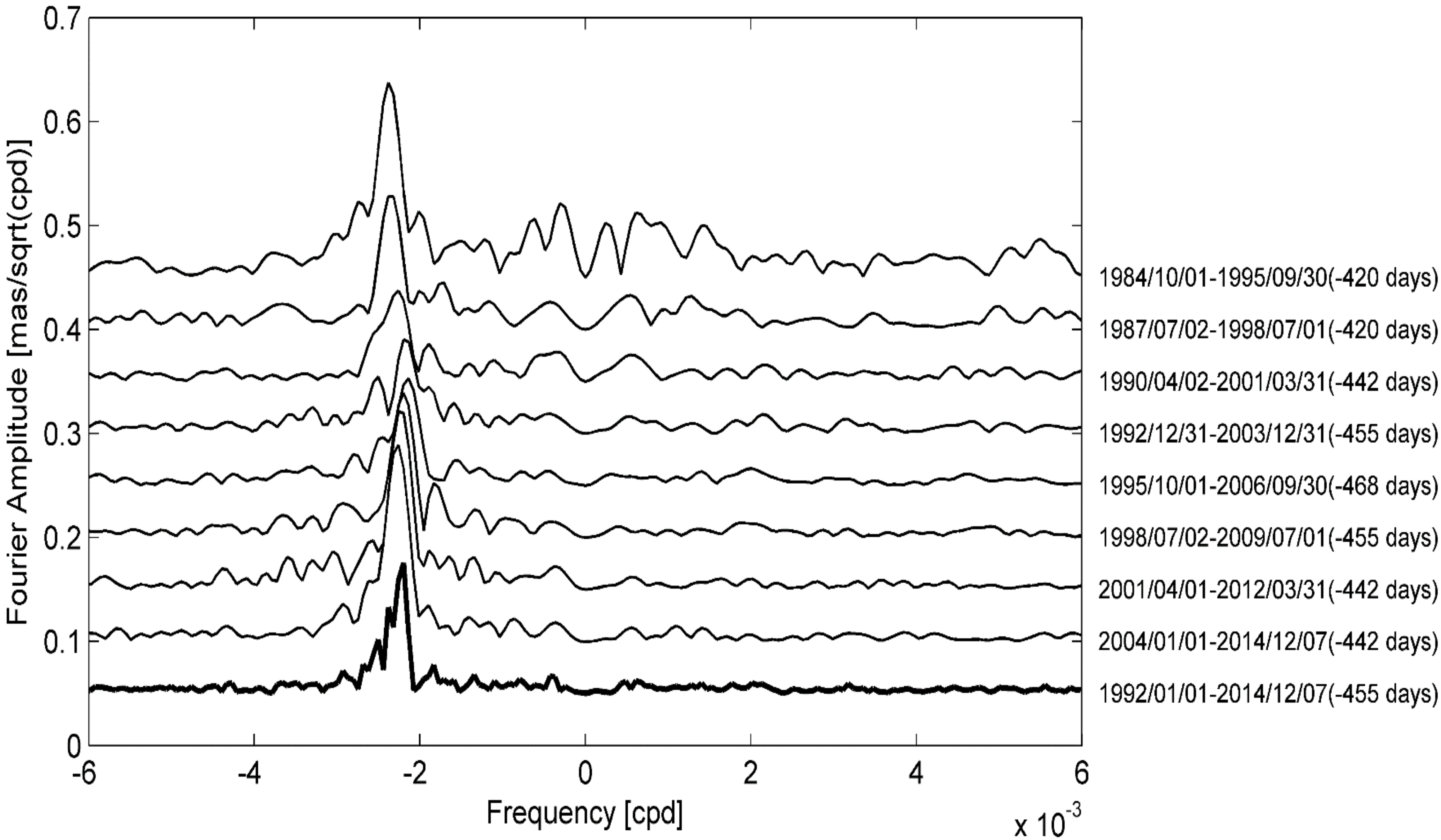


VLBI-observed celestial pole offsets $d\Psi$ and $d\varepsilon$ time series for FCN (courtesy of IERS)

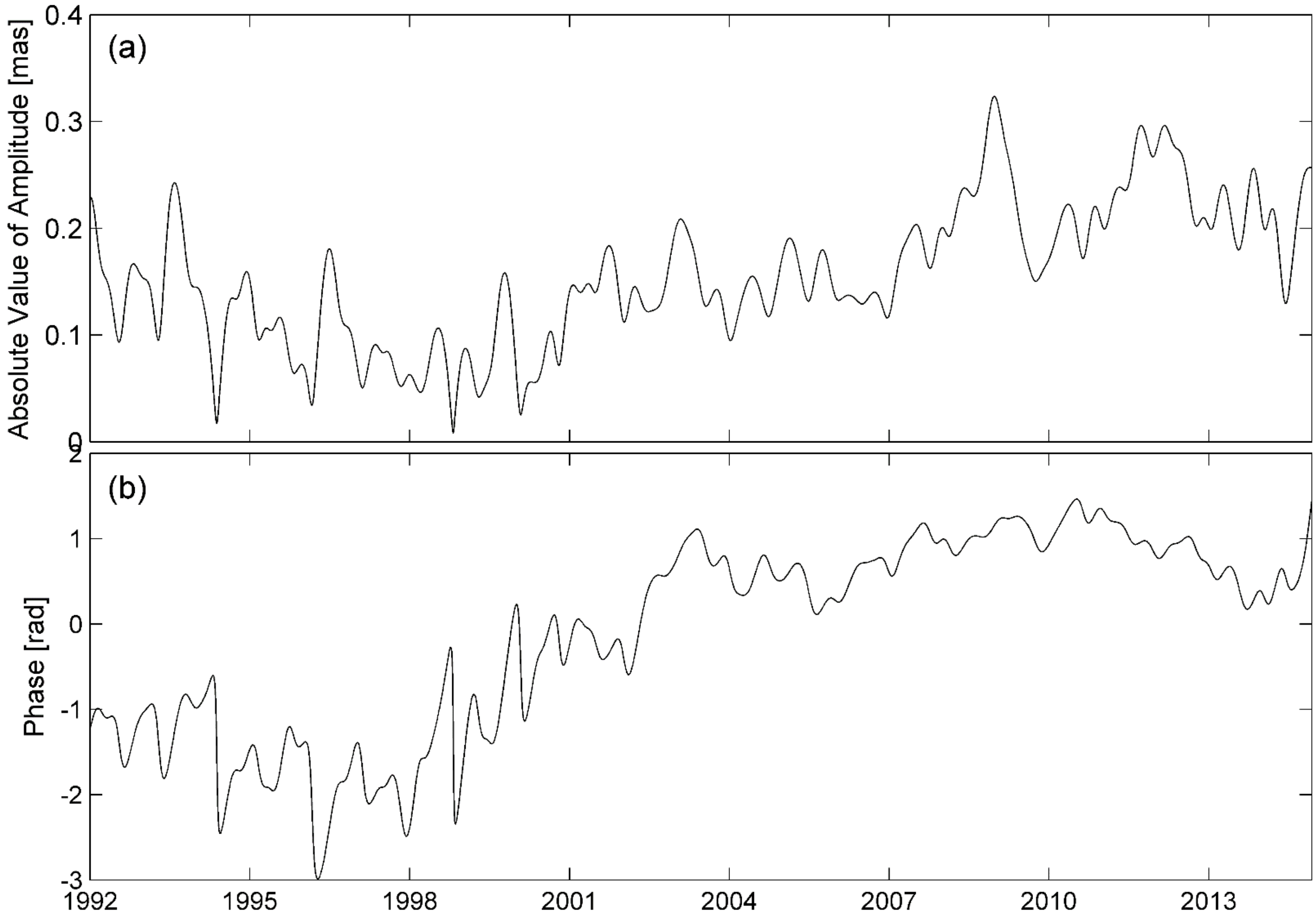
Time-frequency wavelet spectrum of the complex $m(t) = \sin\varepsilon_0 \, d\Psi(t) + i \, d\varepsilon(t)$.

FCN (fluctuating amplitude, apparent period, and phase.)

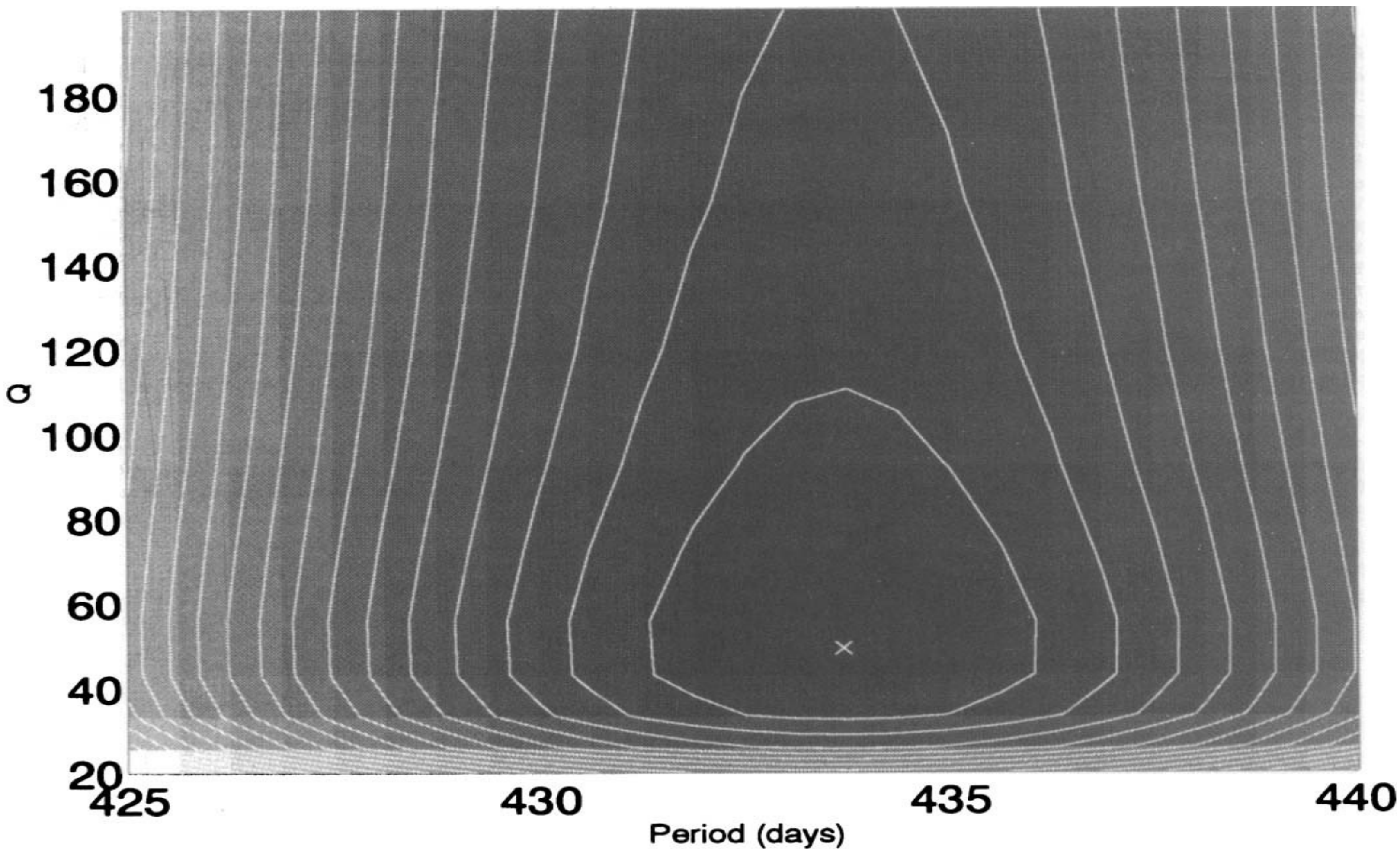
Fluctuating apparent period of FCN



Fluctuating amplitude and phase of FCN

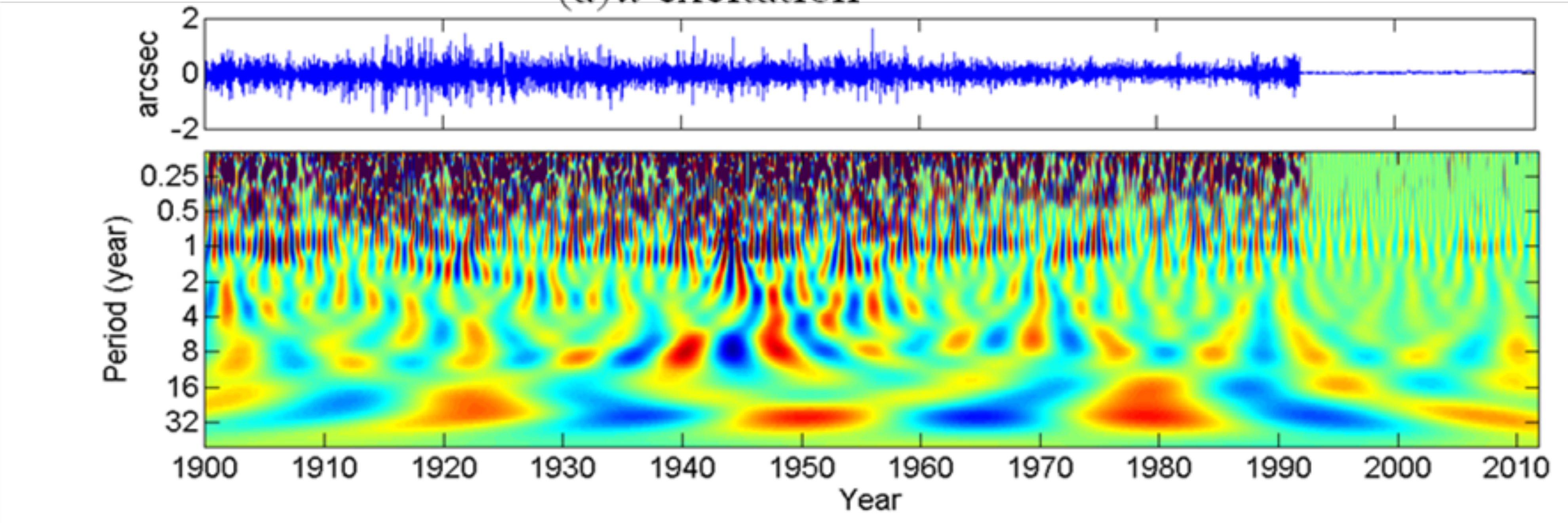


Finding the optimal estimates of Chandler wobble's period P = 433.7 $\pm$1.8 days and quality factor Q = 50 (35,100):

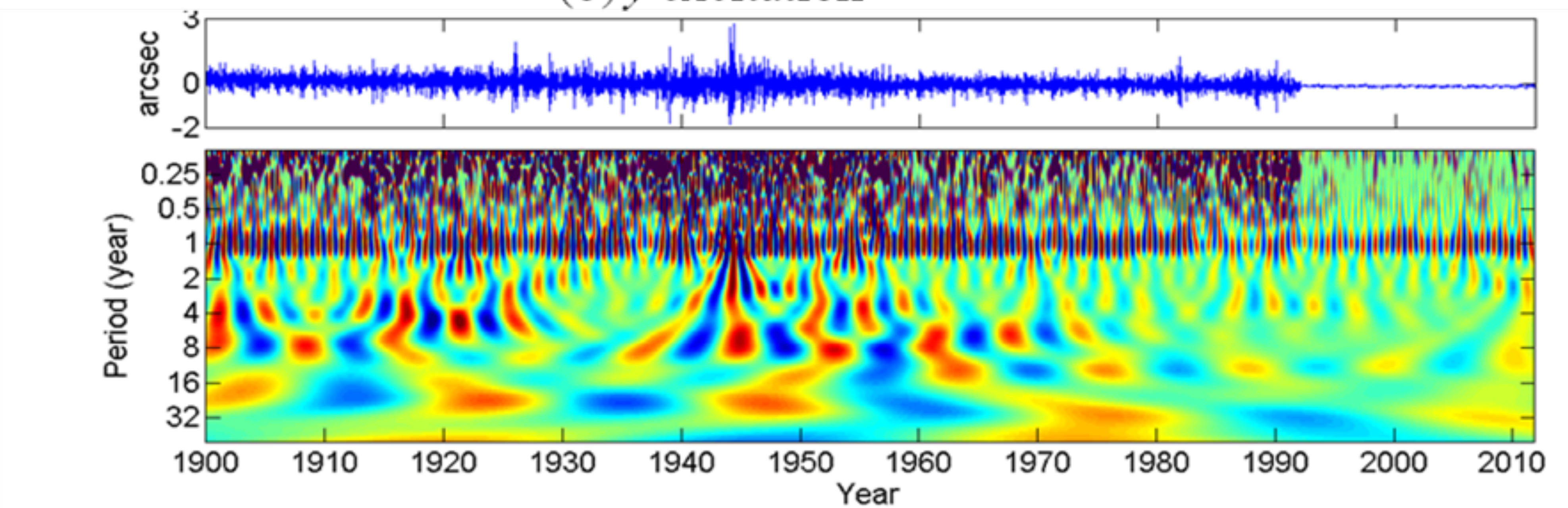


(Furuya and Chao, 1996)

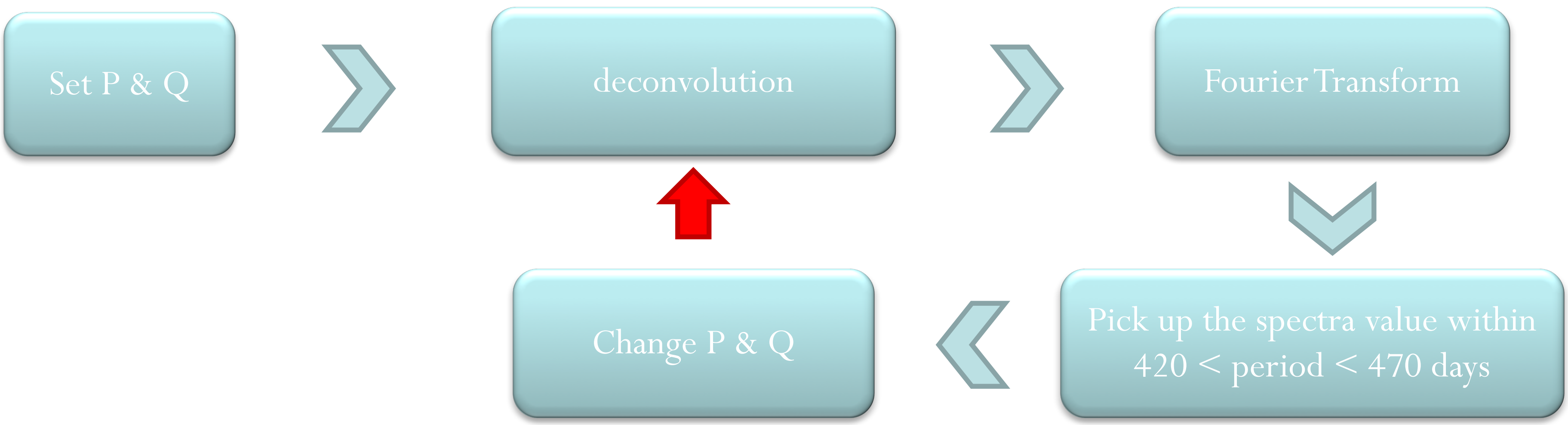
(a) x excitation

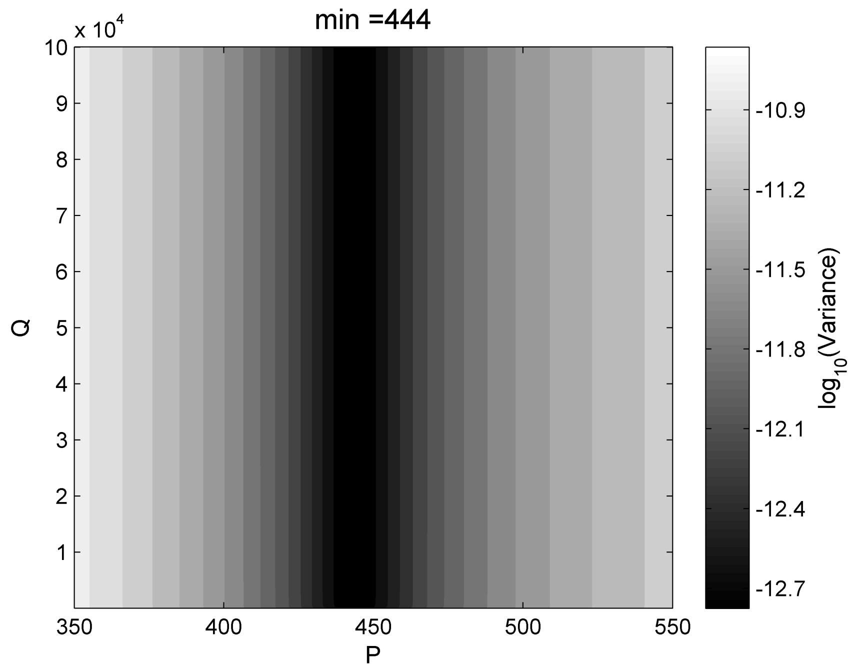


(b) y excitation



Finding minimum excitation after deconvolution of FCN signals

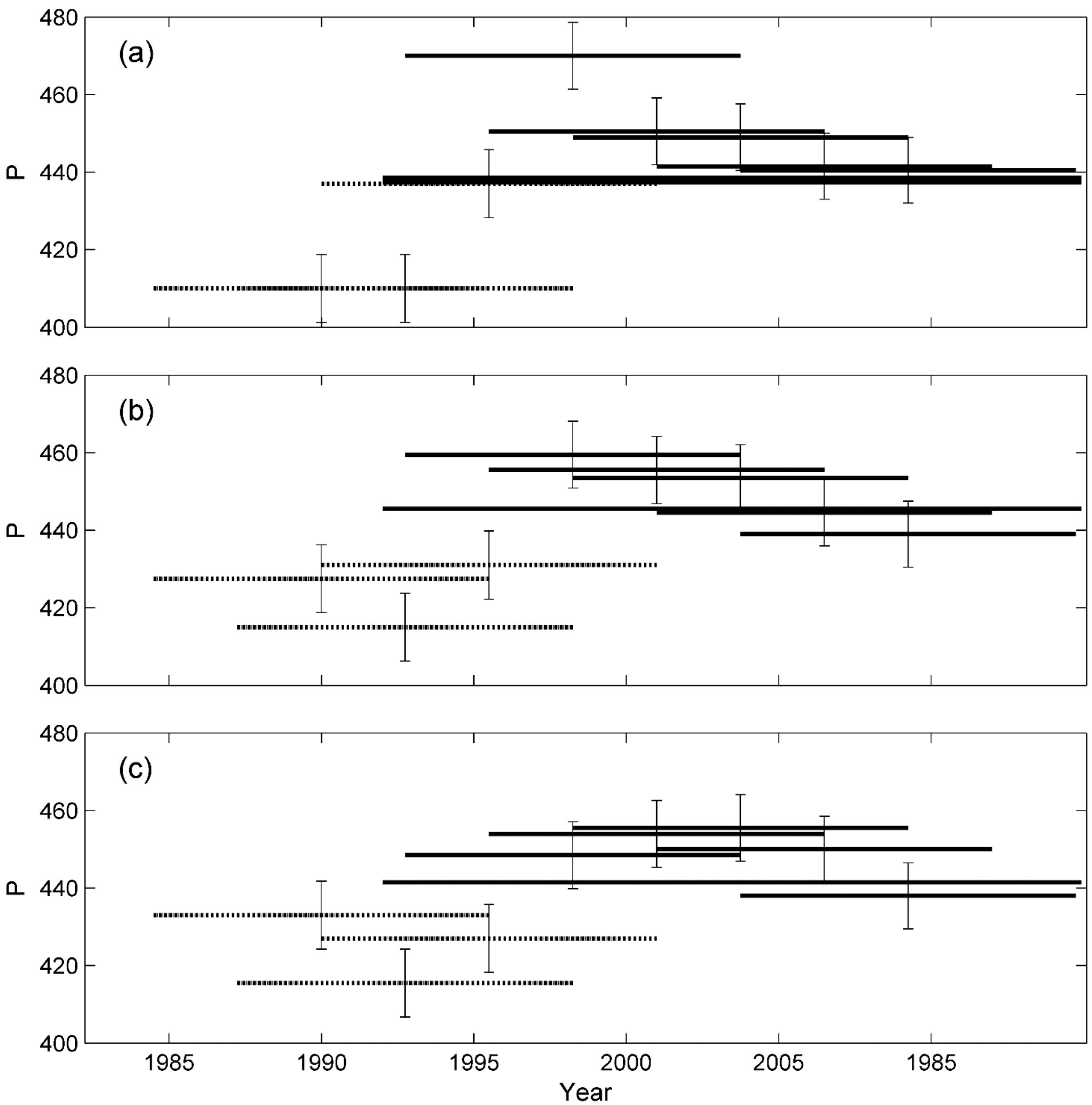




The minimum excitation is found at input *P* of 444 sd, which gives our optimal estimate for *P*, but is completely insensitive to the input *Q* values.

(Chao and Hsieh, 2015)

Deconvolution estimation for FCN period with 1-sigma estimation uncertainty (using 11-year segments of VLBI data as well as the 23-year whole dataset)



applying low-pass filter before deconvolution

applying complex band-pass filter before deconvolution.). We take the 23-year estimate in (b) our "best" final solution.

no filter

(Chao and Hsieh, 2015)

FCN period

Theoretical/model	~460 sd
Resonance/indirect	~430 sd
Deconvolution/direct	~440 sd

The two modes invoke resonance amplification to harmonic oscillations driven at periods close to the natural periods. The resonance transfer function is $1/(\omega - \omega_0)$ (unlike in classical mechanics for non-rotating modes).

Using the resonance of tides, past studies have estimated the natural period of FCN to be 425 - 435 sidereal days, compared to the theoretical value of ~460 sidereal days under hydrostatic-rotational equilibrium.

On the other hand, the VLBI data give the excited FCN time series (directly) since 1980s, in the form of temporal convolution:

$$\int \exp[i\omega_0(t-\tau)]\, Excitation(\tau)\, d\tau$$

So we can perform a deconvolution using an assumed FCN natural period P (and Q) and obtain the corresponding $Excitation(t)$. The one $Excitation(t)$ that has the minimum power (in the relevant frequency band) indicates that the “right” P (and Q) was used, hence an optimal estimate for P (and Q). This scheme is successfully applied for the Chandler wobble (Furuya and Chao, 1996).

Excitation of the Earth's rotational modes as convolution and estimation of period and *Q* using deconvolution

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Harmonic response excited by $\mathbf{E}(\mathbf{r})\exp(i\omega t)$:

$$\mathbf{S}(\mathbf{r},t) = \sum_n \frac{1}{\omega_n^2 - \omega^2} (\mathbf{u}_n^*, \mathbf{E})\, \mathbf{u}_n(\mathbf{r}) \exp(i\omega t)$$

Resonance (in non-rotating frame)

where **E** is the generalized excitation function of space; (,) denotes inner product.

For arbitrary excitation $\mathbf{E}(t)$ (by contour integral):

$$\mathbf{S}(\mathbf{r},t) = \sum_n a_n(t)\, \mathbf{u}_n(\mathbf{r})$$

$$a_n(t) = \frac{1}{\omega_n} \int_{-\infty}^{t} \sin\omega_n (t-\tau)\, E(\tau)\, d\tau$$

NOW LET IT SPIN (Coriolis effect):

Resonance eigenfrequencies ${}_n\omega_{lm}$ shifted

${}_n\omega_{lm}$ split w.r.t. m by "Zeeman-effect" type of break of symmetry

Spatial eigenfunctions modified
-- no longer "normal", but generalized orthogonality:
$(\mathbf{u}_m{*}, \mathbf{u}_n) - 2i\,/(\omega_m+\omega_n)\,(\mathbf{u}_m{*}, \boldsymbol{\Omega}\times\mathbf{u}_n) = \delta_{mn}$

Mode-coupling (selection rules)

Excitation formula modified

4 rotational modes of an oblate (degree-2 order-0) mantle-fluid outer core-solid inner core 3-layer system:

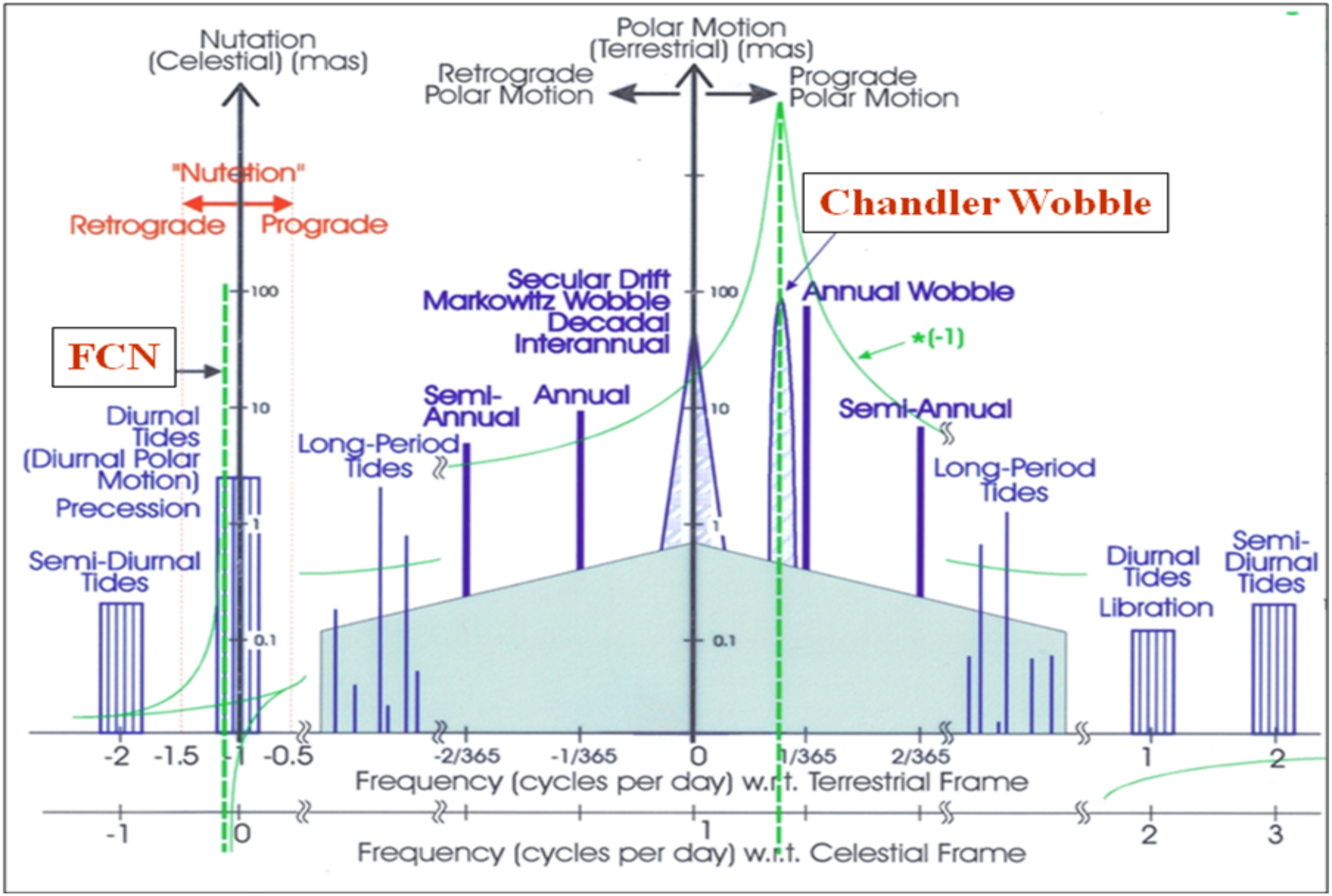
Chandler wobble
Free-core nutation (nearly diurnal free wobble)
Free inner core nutation
Inner core wobble

More rotational modes are possible, for example the torsional modes in fluid outer core, and librational modes under the degree-2 order-2 mantle-inner core gravitational interaction.

The two known rotation modes:

•**Chandler wobble** of the solid mantle has a prograde period in the rotating terrestrial frame of +434 days and a Q of ~50;

•**Free core nutation** of the fluid outer core has a retrograde period in the inertia celestial frame of ~ —4xx sidereal days and a Q TBD.



$m'(t)$ [w.r.t. $\boldsymbol{F}_\Omega$] = $-m(t)$ [w.r.t. $\boldsymbol{F}_0$] $\cdot \exp(-i\Omega t)$.

ω' [w.r.t. $\boldsymbol{F}_\Omega$] = ω [w.r.t. $\boldsymbol{F}_0$] $-\ \Omega$

Normal-mode decomposition
(harmonic response excited by $\mathbf{E}\exp(i\omega t)$):

In non-rotating frame, such as ordinary musical instrument:

$$\mathbf{S}(\mathbf{r},t)=\sum_n \frac{1}{\omega_n^2-\omega^2}(\mathbf{u}_n^*,\mathbf{E})\,\mathbf{u}_n(\mathbf{r})\exp(i\omega t)$$

Resonance (in non-rotating frame)

In rotating frame, such as Erath (with Coriolis force):

$$\mathbf{S}(\mathbf{r},t)=\sum_n \frac{1}{2\omega_n(\omega_n-\omega)}(\mathbf{u}_n^*,\mathbf{E})\,\mathbf{u}_n(\mathbf{r})\exp(i\omega t)$$

Resonance (in rotating frame)

(Chao, 1982)

Excitation of a rotational mode

$$m(t) = \frac{1}{2\Omega(\sigma_0 - \omega)} E \exp(i\omega t) \ , \quad m(t) = x(t) + i\,y(t).$$

for a normal mode excited by a harmonic source in the rotating frame. This equation is the basis for the resonance method of estimating *P* and *Q*.

$$m(t) = -\frac{i}{2\Omega}\int_{-\infty}^{t} \exp[i\sigma_0(t-\tau)]E(\tau)d\tau$$

for an arbitrary excitation $E(t)$. This equation of convolution describes the observed motion – fluctuating amplitude, apparent period, and phase.

$$\frac{i}{\sigma_0}\frac{dm(t)}{dt} + m(t) = \frac{1}{2\Omega\sigma_0}E(t)$$

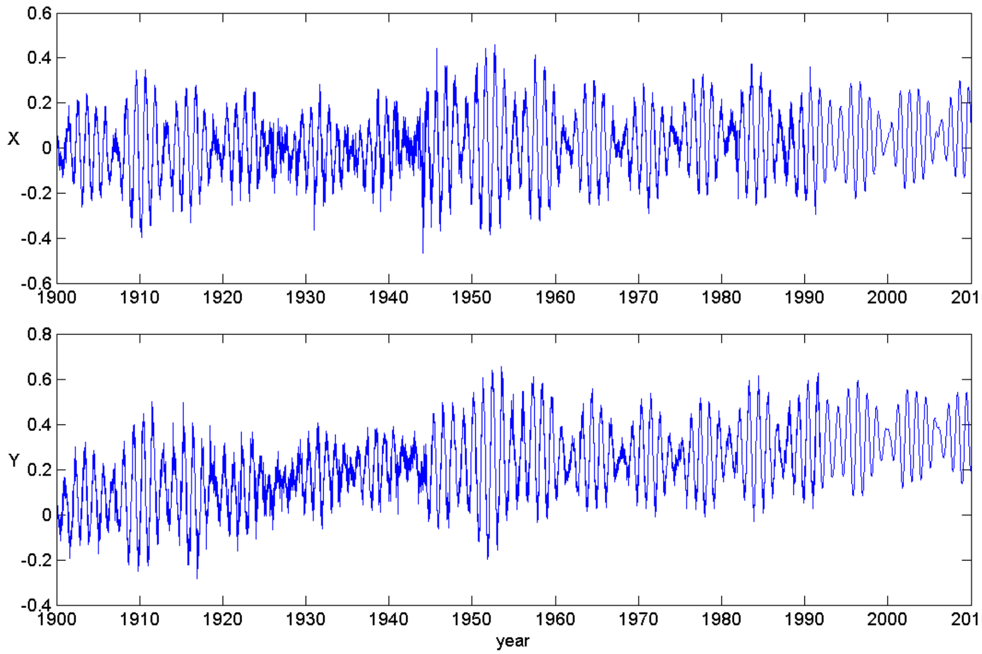
as a differential deconvolution form of the excitation equation. This equation is the basis for the deconvolution method of estimating *P* and *Q*.

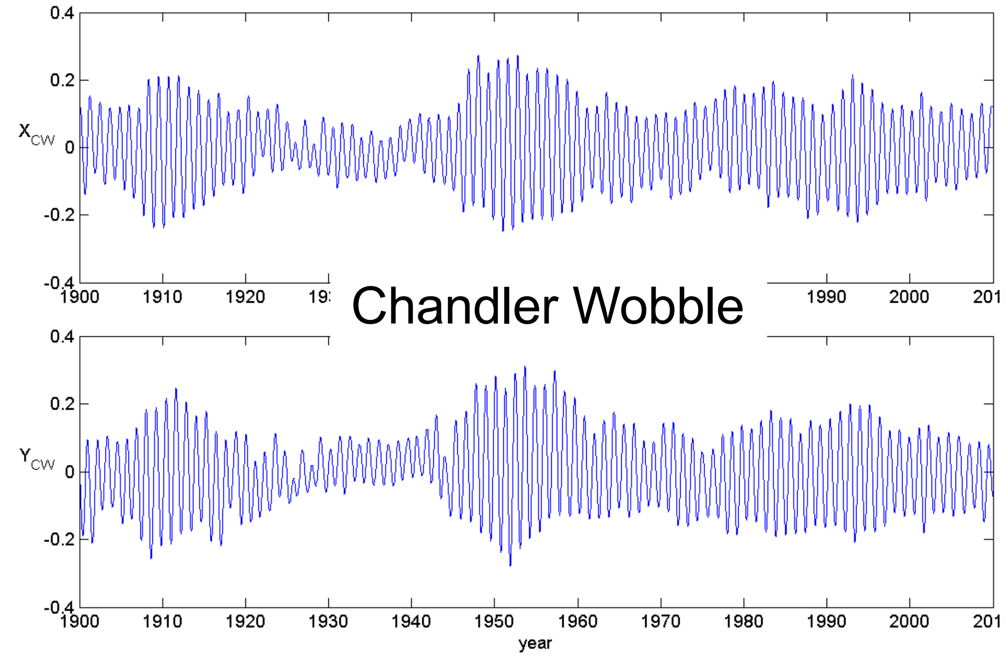
Convolution:

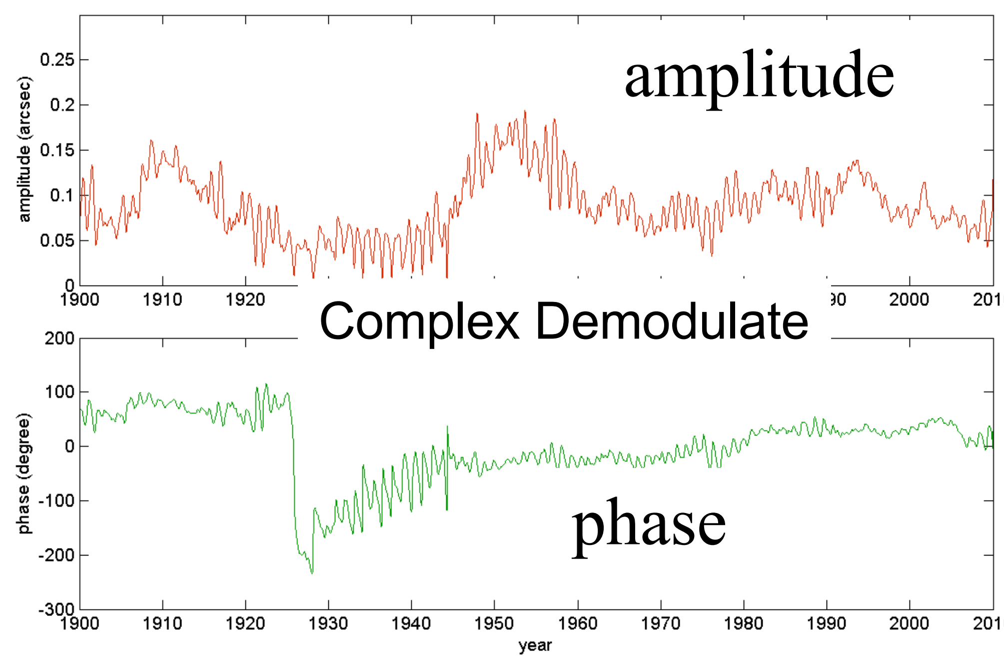
Resonance upon continual excitation by broad-band source

(i)The oscillation would in general appear fairly smooth and quasi-periodic.
(ii) The apparent period is never far from the natural resonance period but would fluctuate somewhat with time because of (iii) below.
(iii) The apparent amplitude would undulate with time by the action of excitation, far from just due to the exponential damping.
(iv)The apparent phase would fluctuate with time, but kept more or less steady except during (v) below.
(v)The phase would undergo significant changes whenever the amplitude becomes near-zero.

Polar Motion



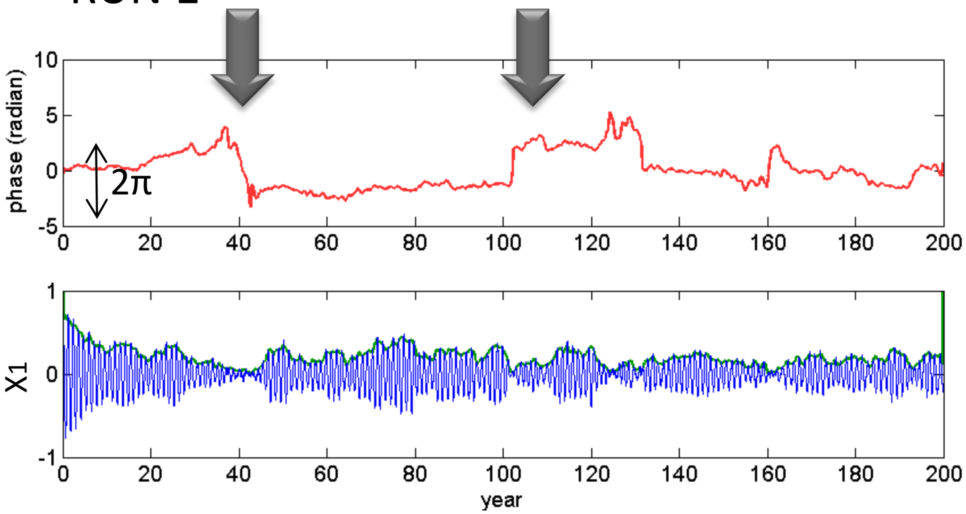




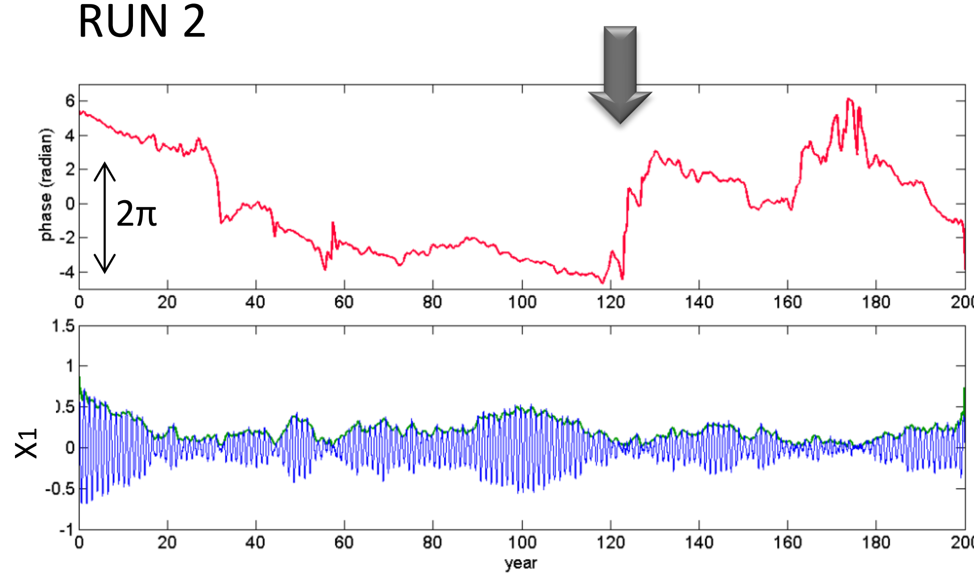
Chao & Chung (2012)

Monte Carlo experiments

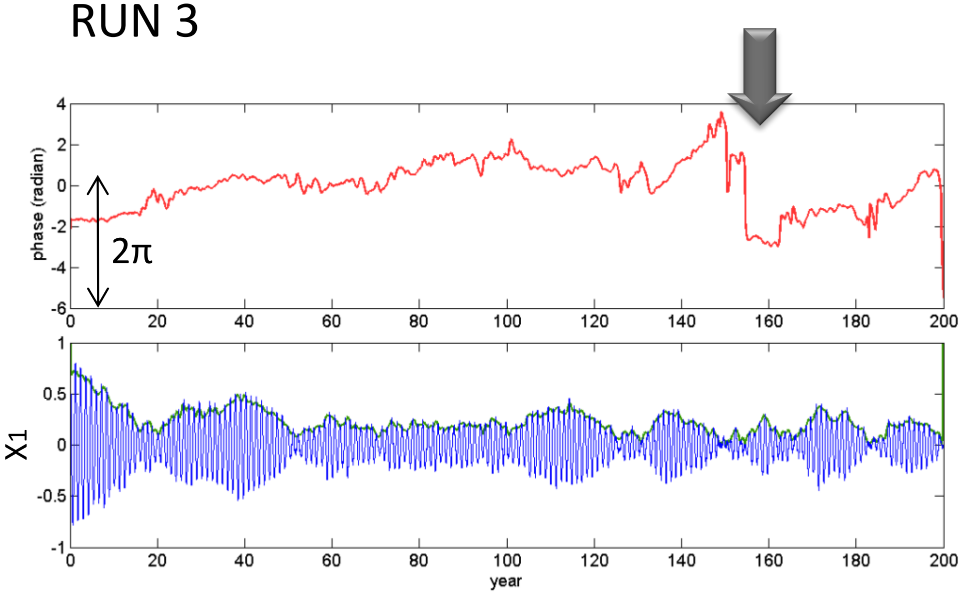
RUN 1



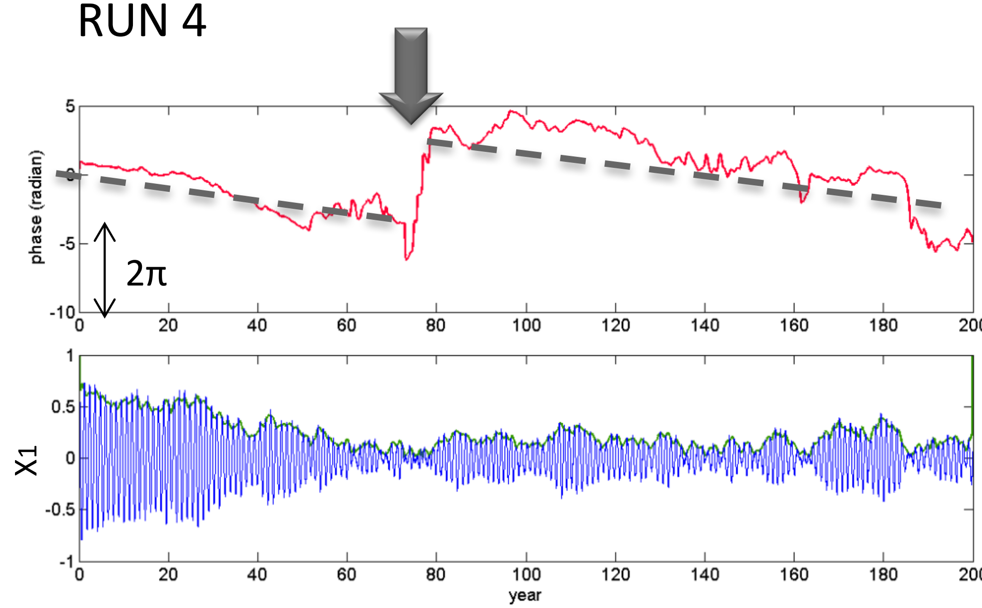
RUN 2



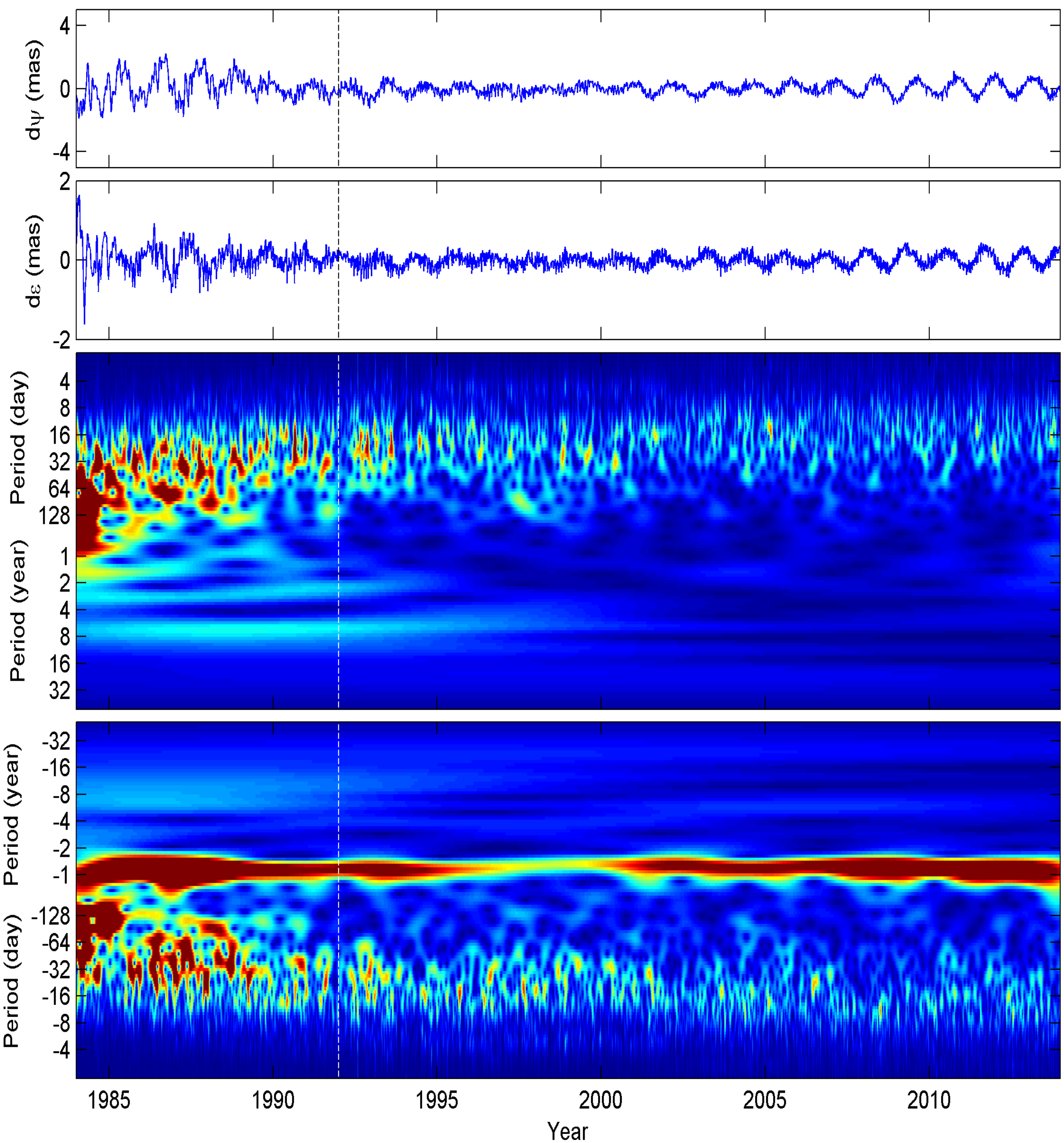
RUN 3



RUN 4



Chao & Chung (2012)



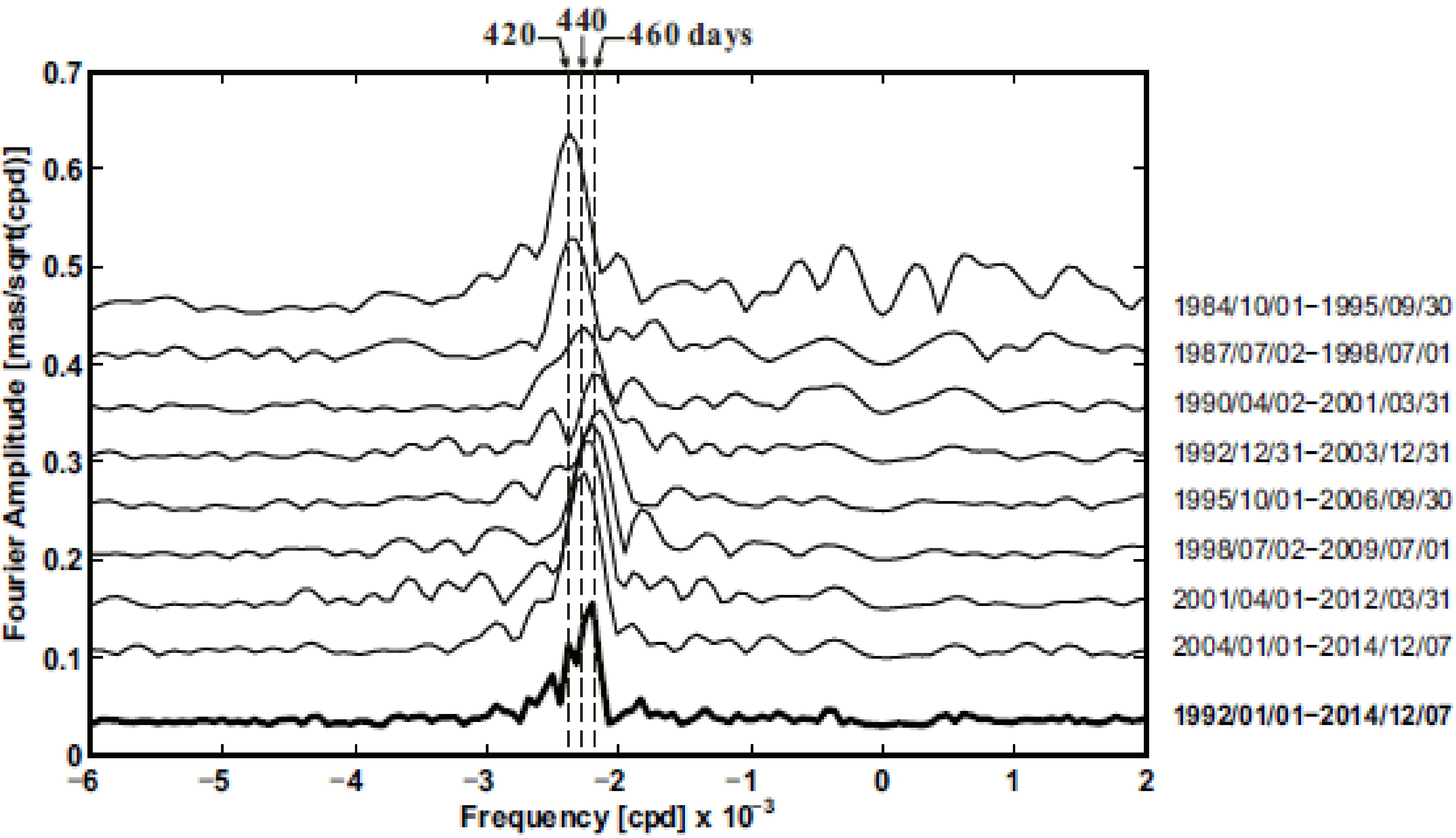
VLBI-observed celestial pole offsets $d\Psi$ and $d\varepsilon$ time series for FCN (courtesy of IERS)

Time-frequency wavelet spectrum of the complex $m(t) = \sin\varepsilon_0\, d\Psi(t) + i\, d\varepsilon(t)$.

FCN (fluctuating amplitude, apparent period, and phase.)

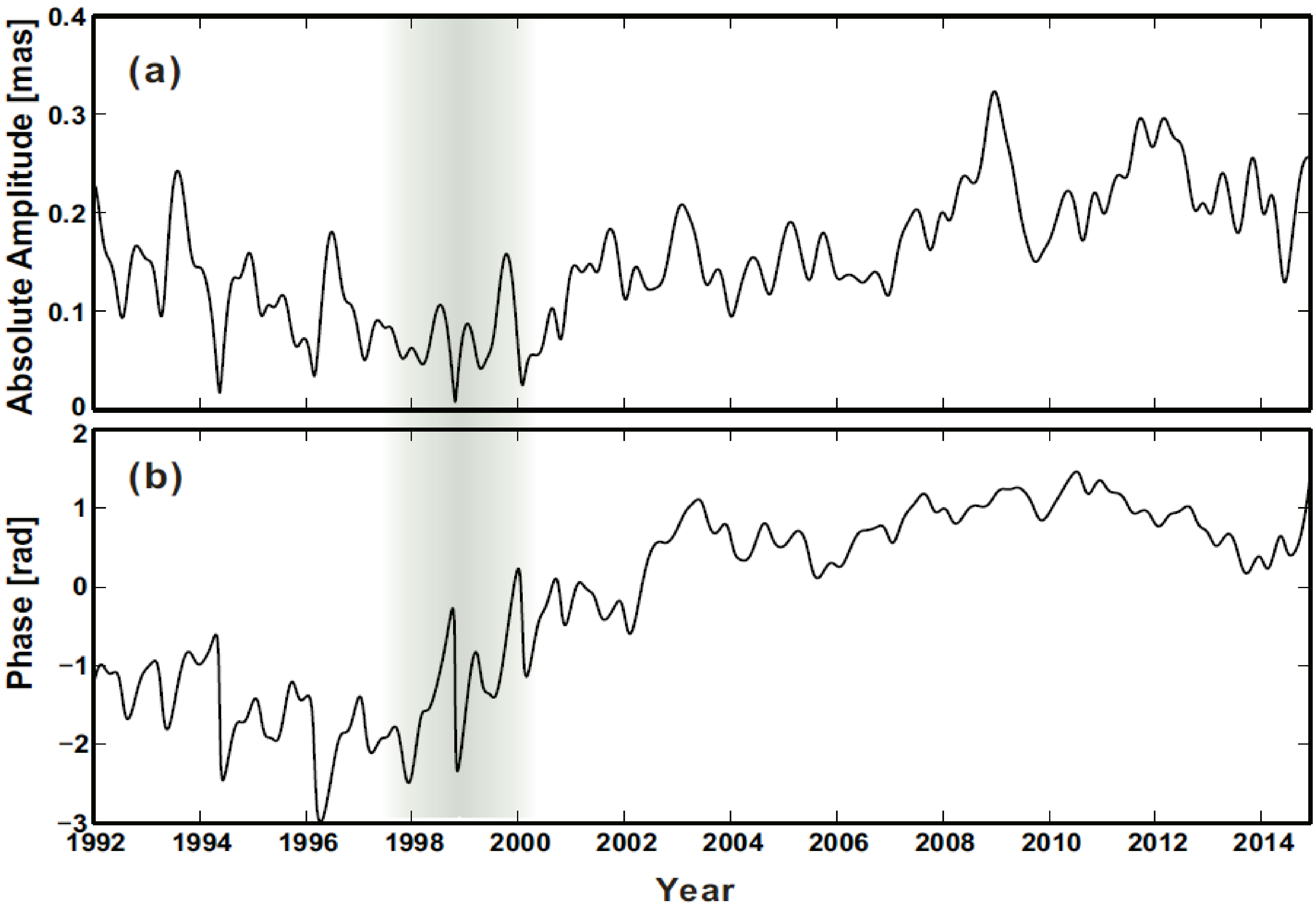
Chao & Hsieh (2015)

Fluctuating apparent period of FCN



Chao & Hsieh (2015)

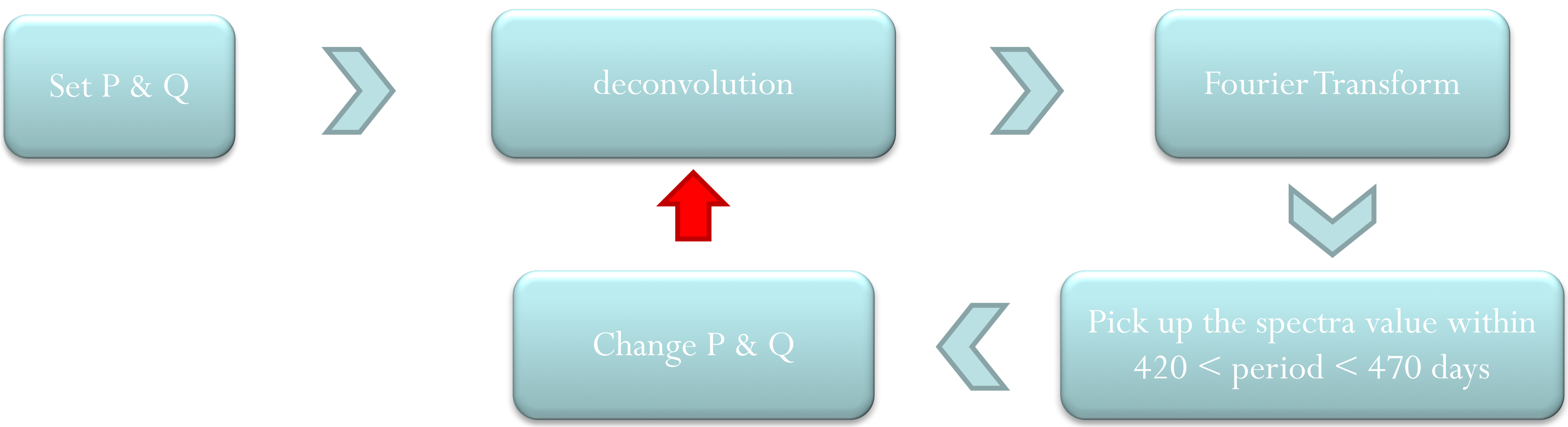
Fluctuating amplitude and phase of FCN (using complex demodulation)



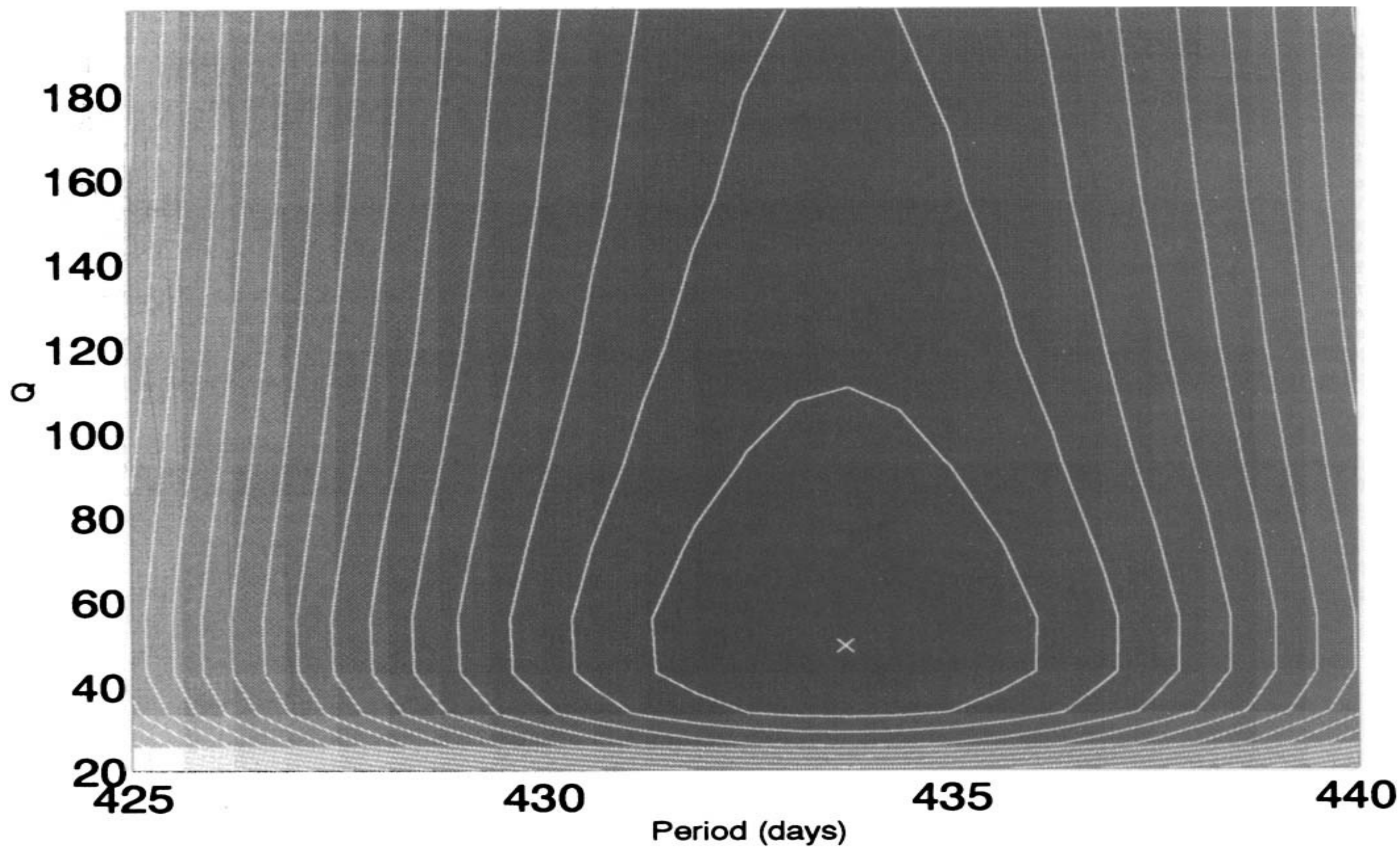
Chao & Hsieh (2015)

Estimating the period (P) and quality factor (Q) using Deconvolution

-- searching the (P,Q) space for the minimum power of the deconvolved excitation

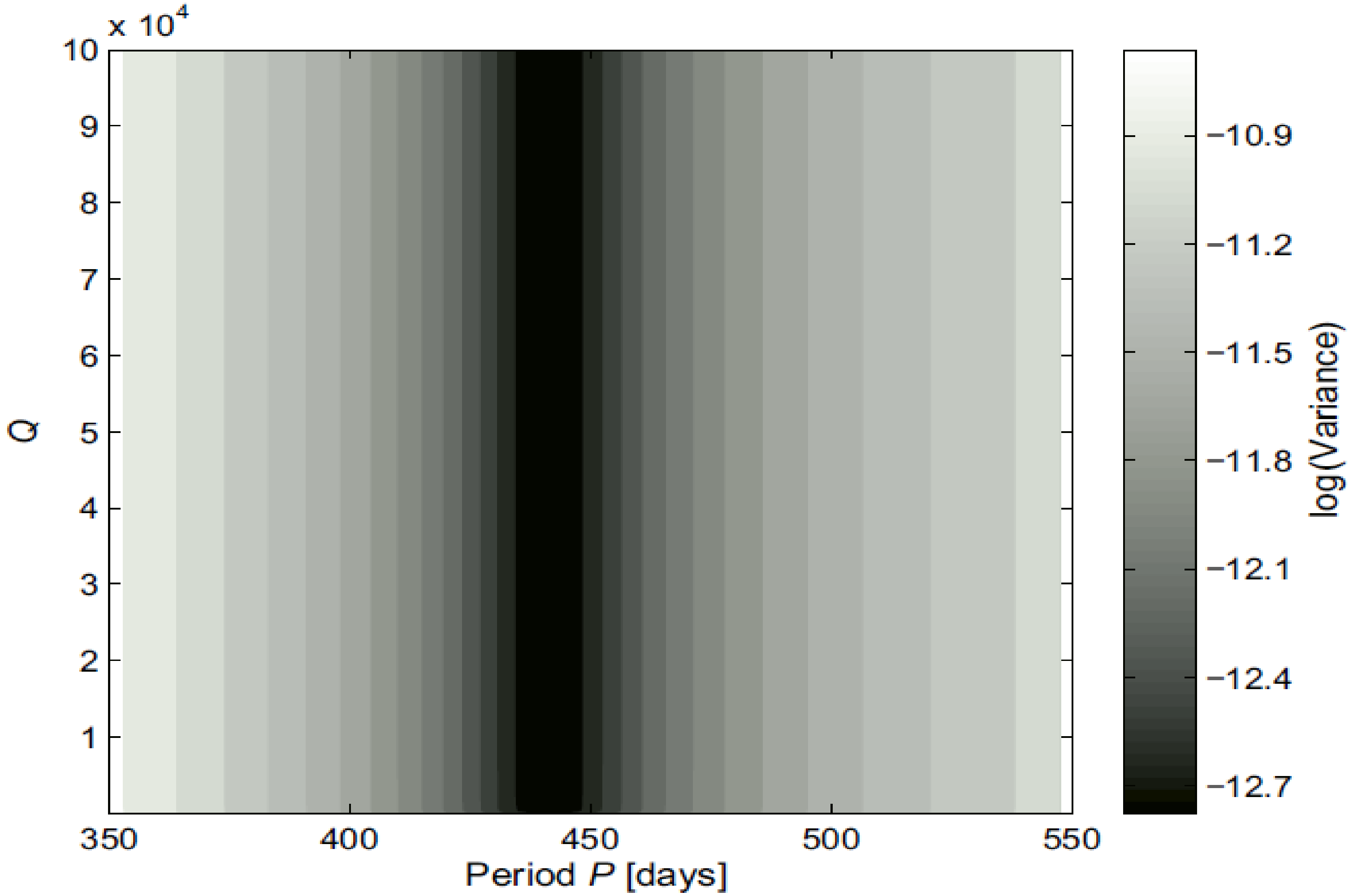


Finding minimum excitation in deconvolution of Chandler wobble
=> $P = 433.7 \pm 1.8$ days, $Q = 50$ (35,100)



Furuya & Chao (1996)

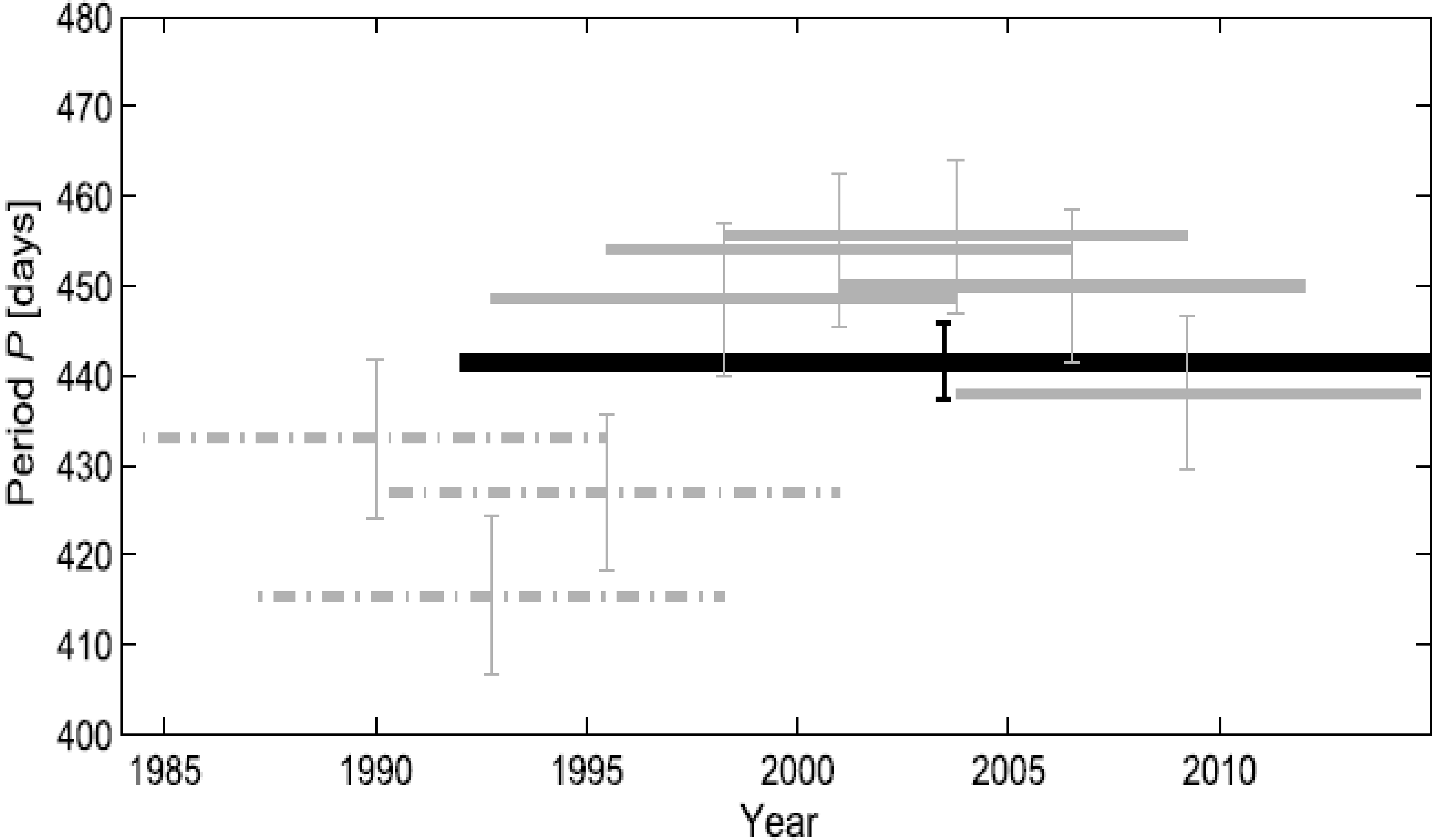
Finding minimum excitation in deconvolution of FCN



The minimum excitation is found at input *P* of 441 sd, which gives our optimal estimate for *P*, but is completely insensitive to the input *Q* values.

Chao & Hsieh (2015)

Deconvolution estimation for FCN period with 1-sigma estimation uncertainty (441 $\pm$ 4.5 sd)



Chao & Hsieh (2015)

FCN period (sidereal days)

Theoretical/model	460 sd
Resonance/indirect	425 - 435 sd
Deconvolution/direct	441 $\pm$ 4.5 sd